

First-Order Logic

Part One

Recap from Last Time

Recap So Far

- A ***propositional variable*** is a variable that is either true or false.
- The ***propositional connectives*** are as follows:
 - Negation: $\neg p$
 - Conjunction: $p \wedge q$
 - Disjunction: $p \vee q$
 - Implication: $p \rightarrow q$
 - Biconditional: $p \leftrightarrow q$
 - True: \top
 - False: \perp

Take out a sheet of paper!

What's the truth table for the \rightarrow connective?

What's the negation of $p \rightarrow q$?

New Stuff!

First-Order Logic

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
 - ***predicates*** that describe properties of objects,
 - ***functions*** that map objects to one another, and
 - ***quantifiers*** that allow us to reason about multiple objects.

Some Examples

Likes(You, Eggs) \wedge Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)



Likes(You, Eggs) \wedge Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)

Learns(You, History) \vee ForeverRepeats(You, History)

In(MyHeart, Havana) \wedge TookBackTo(Him, Me, EastAtlanta)

Likes(You, Eggs) ∧ Likes(You, Tomato) → Likes(You, Shakshuka)

Learns(You, History) ∨ ForeverRepeats(You, History)

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These blue terms are called *constant symbols*. Unlike propositional variables, they refer to objects, not propositions.

Likes(You, Eggs) ∧ Likes(You, Tomato) → Likes(You, Shakshuka)

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The red things that look like function calls are called *predicates*.
Predicates take objects as arguments and evaluate to true or false.

Likes(You, Eggs) ∧ Likes(You, Tomato) → Likes(You, Shakshuka)

Learns(You, History) ∨ ForeverRepeats(You, History)

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What remains are traditional propositional connectives. Because each predicate evaluates to true or false, we can connect the truth values of predicates using normal propositional connectives.

Reasoning about Objects

- To reason about objects, first-order logic uses ***predicates***.
- Examples:

Cute(Quokka)

Cool(CS103 students)

- Applying a predicate to arguments produces a proposition, which is either true or false.
- Typically, when you're working in FOL, you'll have a list of predicates, what they stand for, and how many arguments they take. It'll be given separately from the formulas you write.

First-Order Sentences

- Sentences in first-order logic can be constructed from predicates applied to objects:

$Cute(a) \rightarrow Dikdik(a) \vee Kitty(a) \vee Puppy(a)$

$Succeeds(You) \leftrightarrow Practices(You)$

$x < 8 \rightarrow x < 137$

The less-than sign is just another predicate. Binary predicates are sometimes written in *infix notation* this

Numbers are not "built in" to first-order logic. They're constant symbols just like "You" and "a" above.

Equality

- First-order logic is equipped with a special predicate $=$ that says whether two objects are equal to one another.
- Equality is a part of first-order logic, just as \rightarrow and \neg are.
- Examples:
 - MilesMorales = SpiderMan*
 - MorningStar = EveningStar*
- Equality can only be applied to **objects**; to state that two **propositions** are equal, use \leftrightarrow .

Let's see some more examples.

*FavoriteMovieOf(You) ≠ FavoriteMovieOf(Date) ∧
StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))*

*FavoriteMovieOf(You) ≠ FavoriteMovieOf(Date) ∧
StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))*

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*FavoriteMovieOf(You) ≠ FavoriteMovieOf(Date) ∧
StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))*

These purple terms are *functions*. Functions take objects as input and produce objects as output.

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StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))*

*FavoriteMovieOf(You) ≠ FavoriteMovieOf(Date) ∧
StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))*

*FavoriteMovieOf(You) ≠ FavoriteMovieOf(Date) ∧
StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))*

Functions

- First-order logic allows **functions** that return objects associated with other objects.
- Examples:

ColorOf(Money)

MedianOf(x, y, z)

$x + y$

- As with predicates, functions can take in any number of arguments, but always return a single value.
- Functions evaluate to **objects**, not **propositions**.

Objects and Predicates

- When working in first-order logic, be careful to keep objects (actual things) and propositions (true or false) separate.

- You cannot apply connectives to objects:



Venus \rightarrow *TheSun*



- You cannot apply functions to propositions:



StarOf(IsRed(Sun) \wedge IsGreen(Mars))



- Ever get confused? *Just ask!*

The Type-Checking Table

	... operate on and produce
Connectives (\leftrightarrow , \wedge , etc.) ...	propositions	a proposition
Predicates ($=$, etc.) ...	objects	a proposition
Functions ...	objects	an object

One last (and major) change

Some spider is radioactive.

Some spider is radioactive.

$\exists s. (Spider(s) \wedge Radioactive(s))$

Some spider is radioactive.

$\exists s. (Spider(s) \wedge Radioactive(s))$

\exists is the **existential quantifier**
and says "for some choice
of s , the following is true."

The Existential Quantifier

- A statement of the form

$\exists x.$ *some-formula*

is true if there exists a choice of x where ***some-formula*** is true when that x is plugged into it.

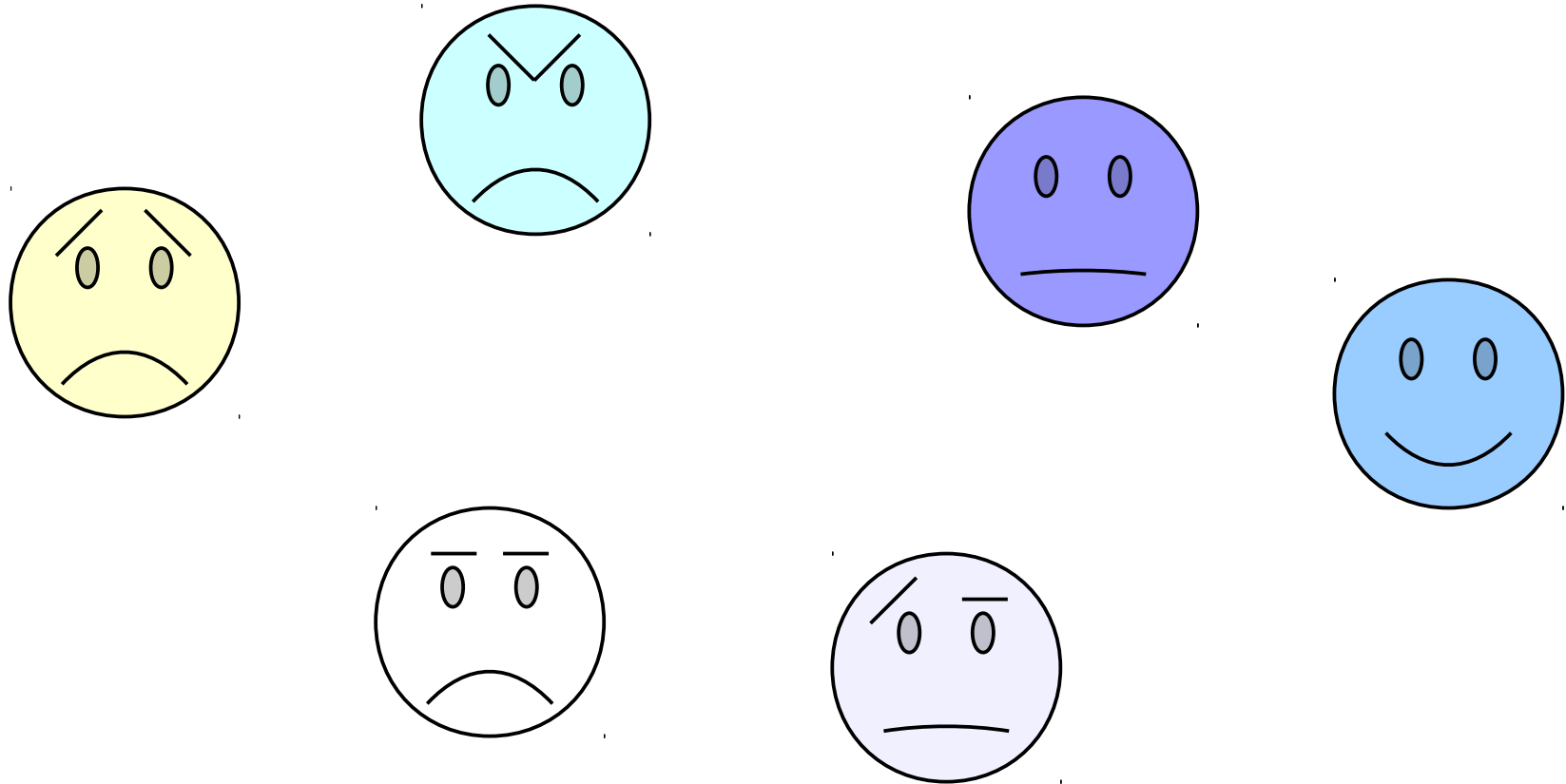
- Examples:

$\exists x. (Even(x) \wedge Prime(x))$

$\exists x. (TallerThan(x, me) \wedge LighterThan(x, me))$

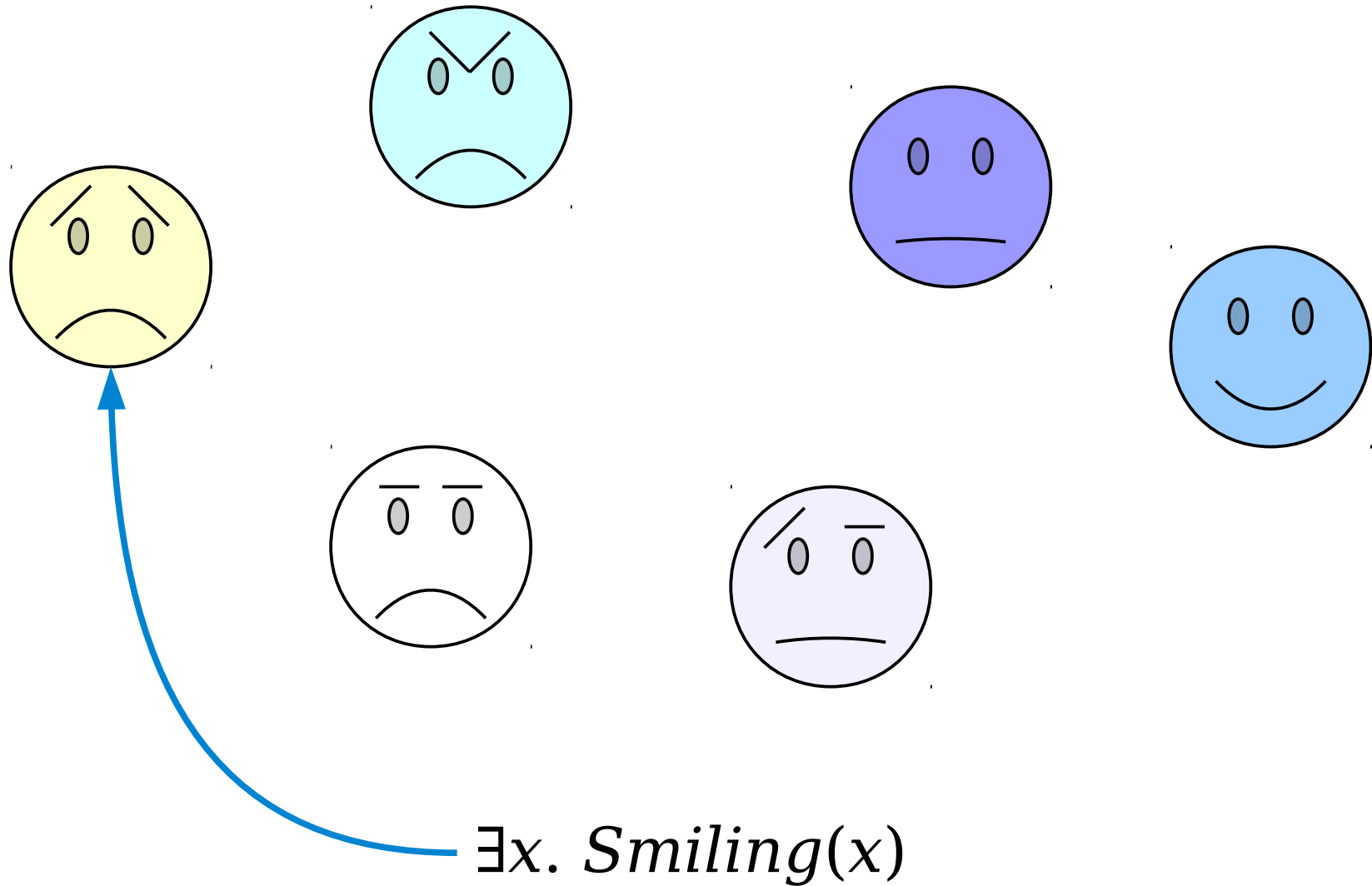
$(\exists w. Will(w)) \rightarrow (\exists x. Way(x))$

The Existential Quantifier

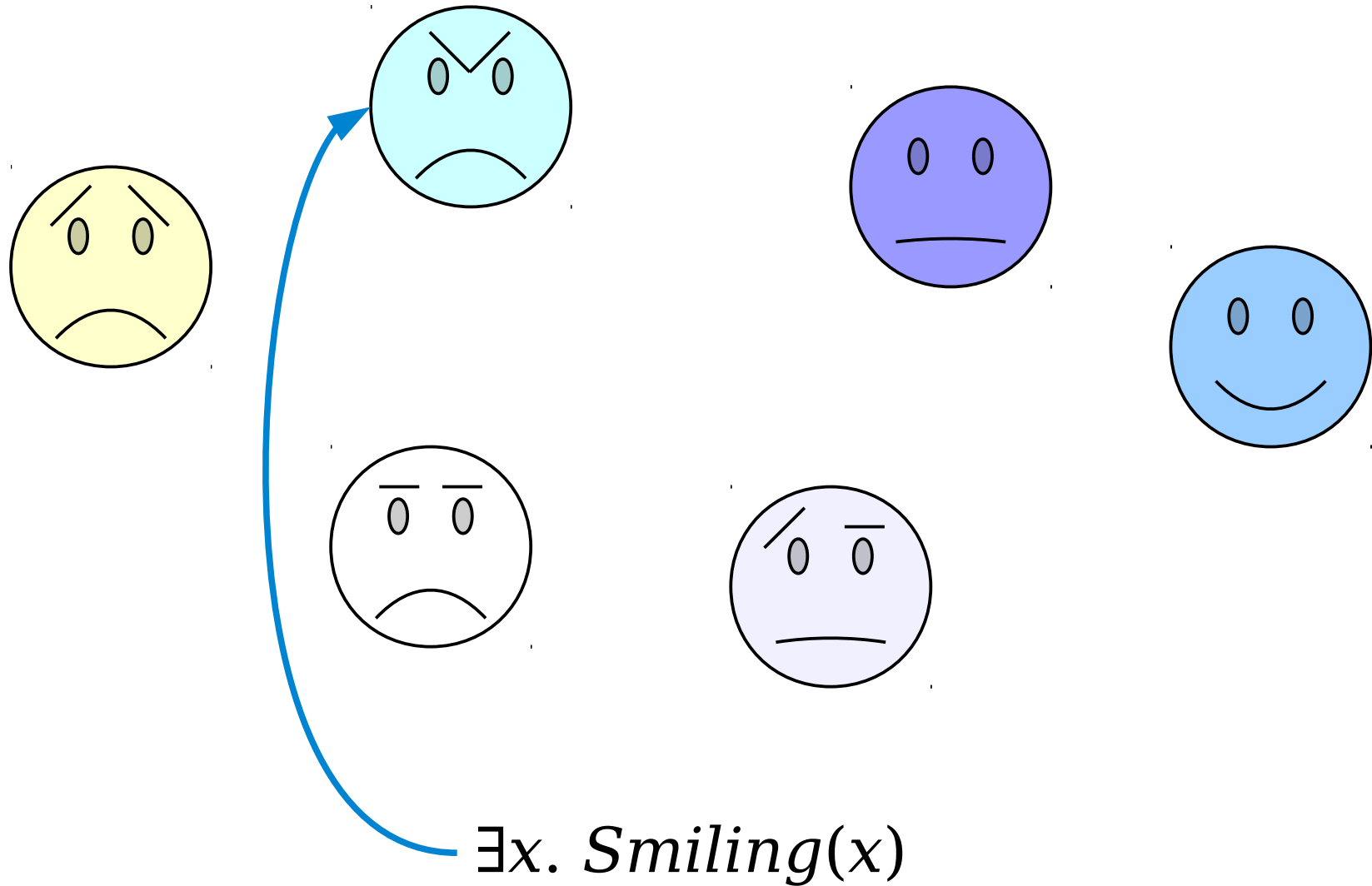


$\exists x. \textit{Smiling}(x)$

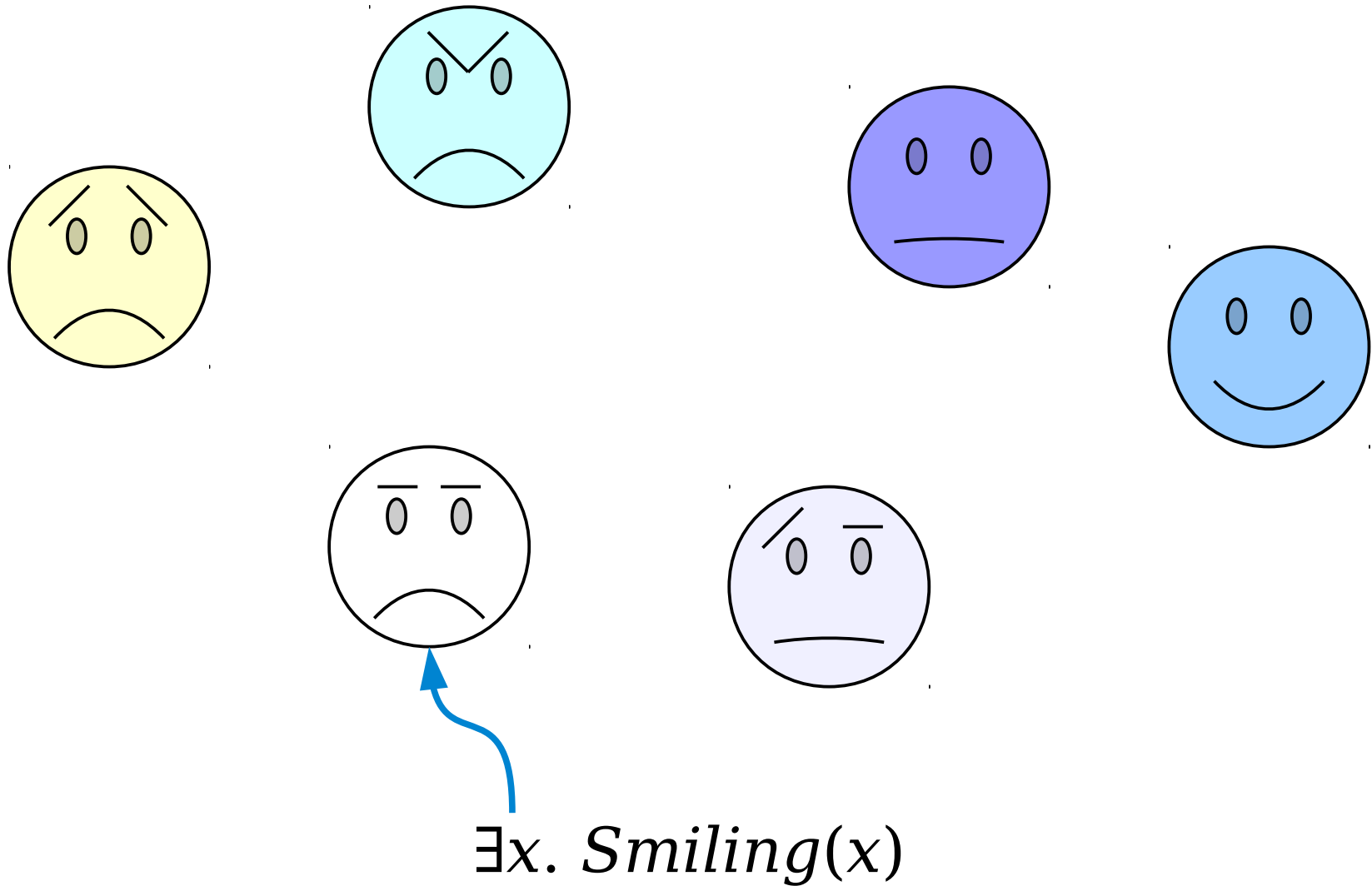
The Existential Quantifier



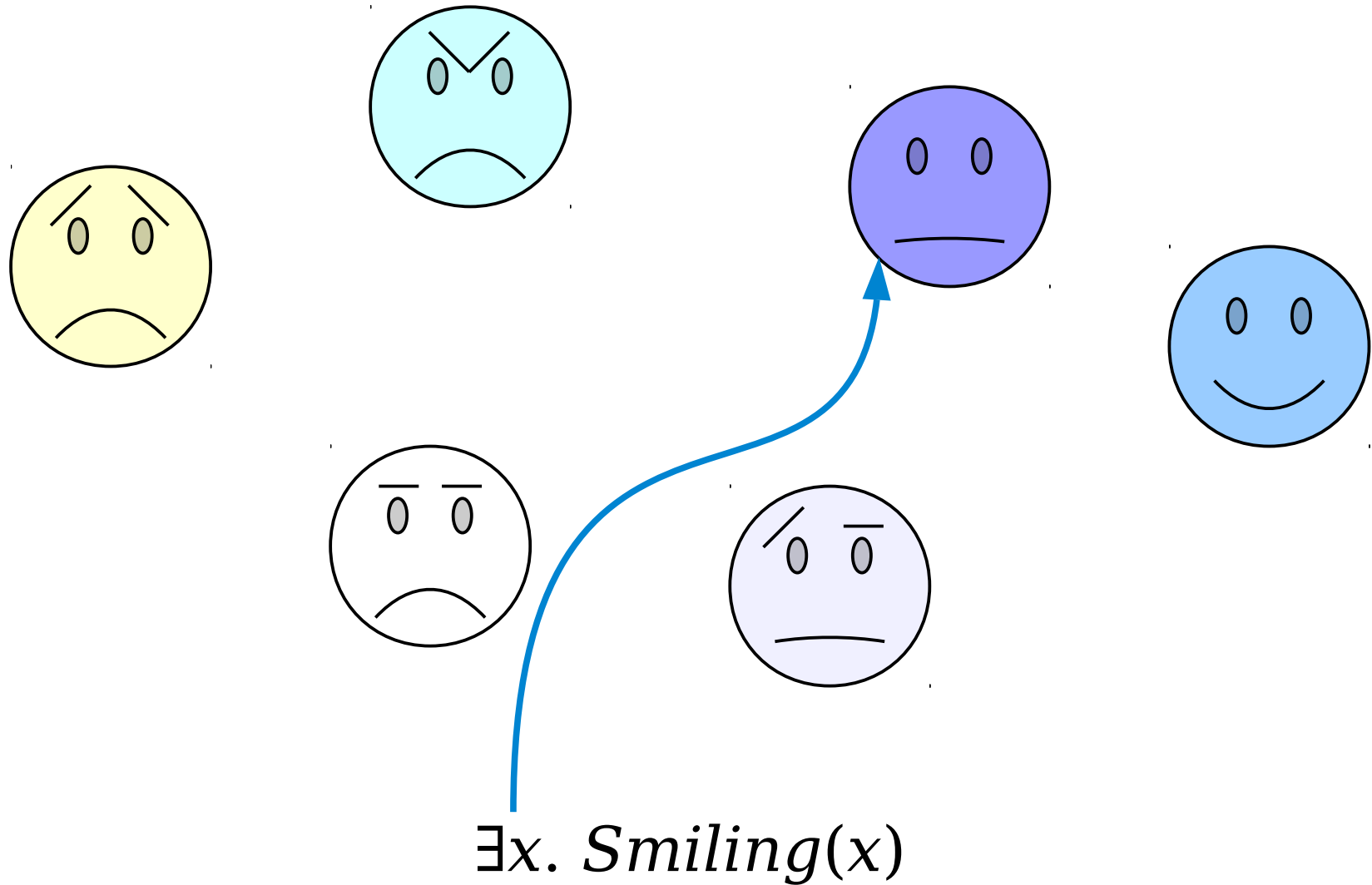
The Existential Quantifier



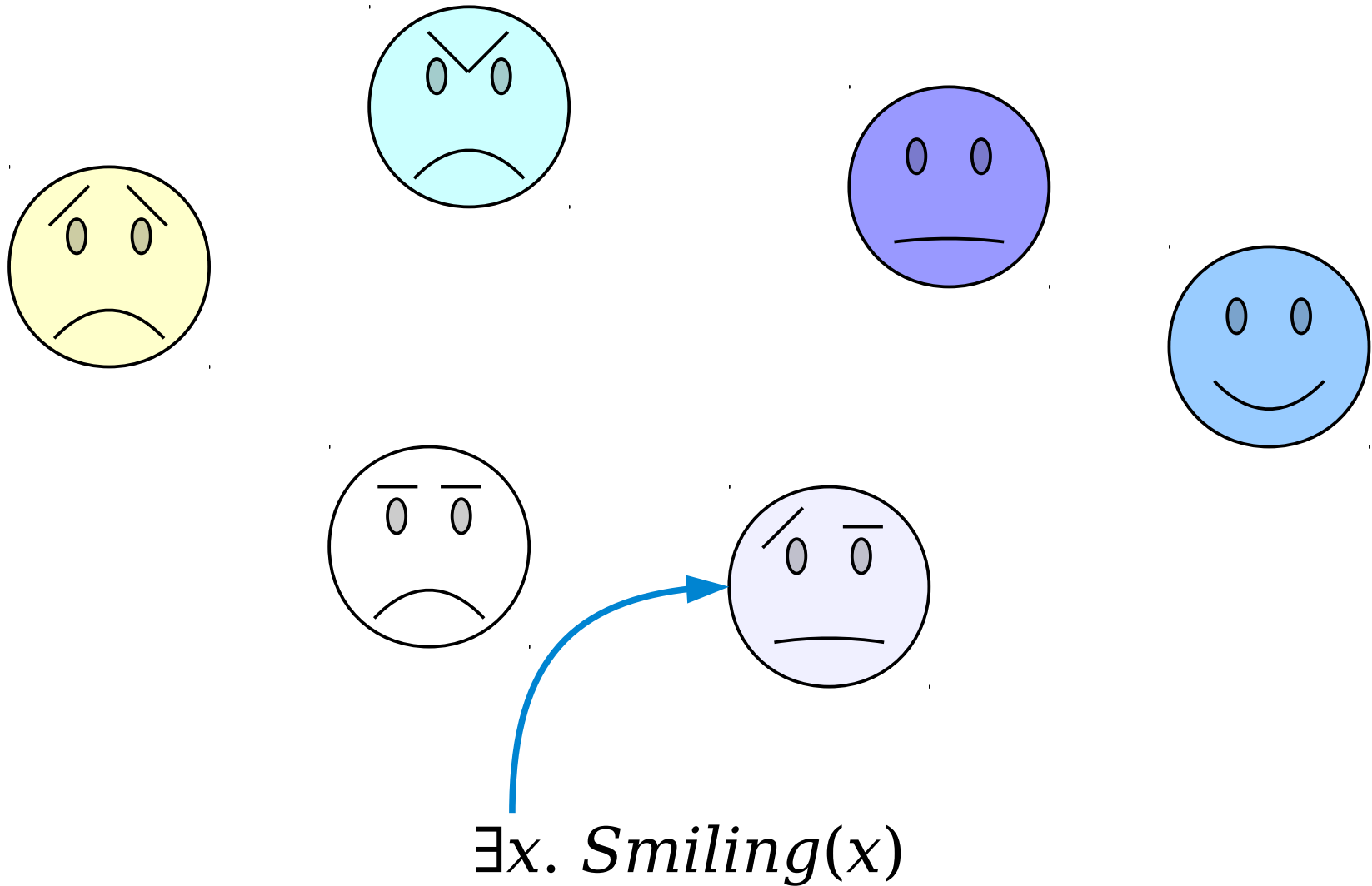
The Existential Quantifier



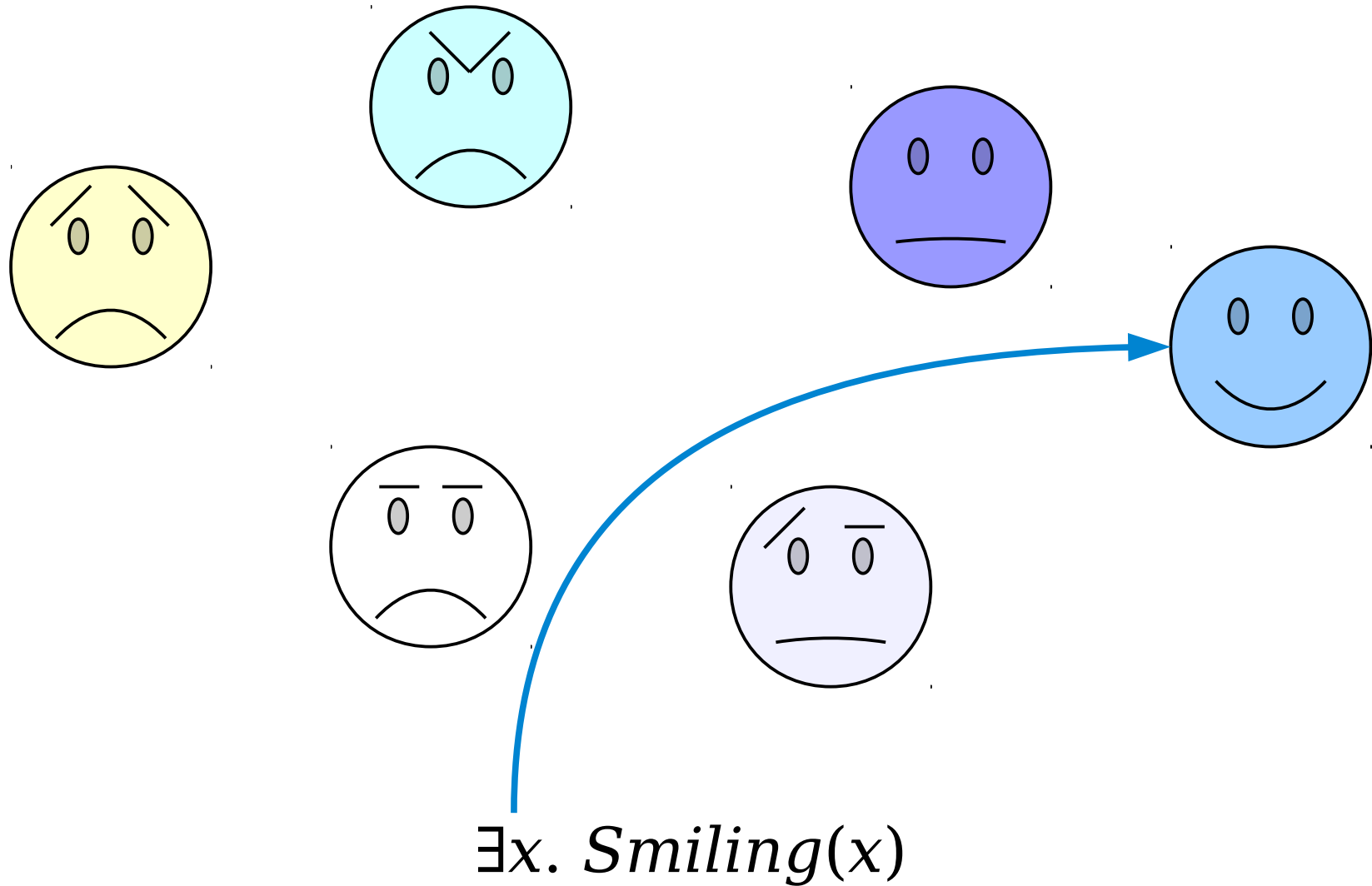
The Existential Quantifier



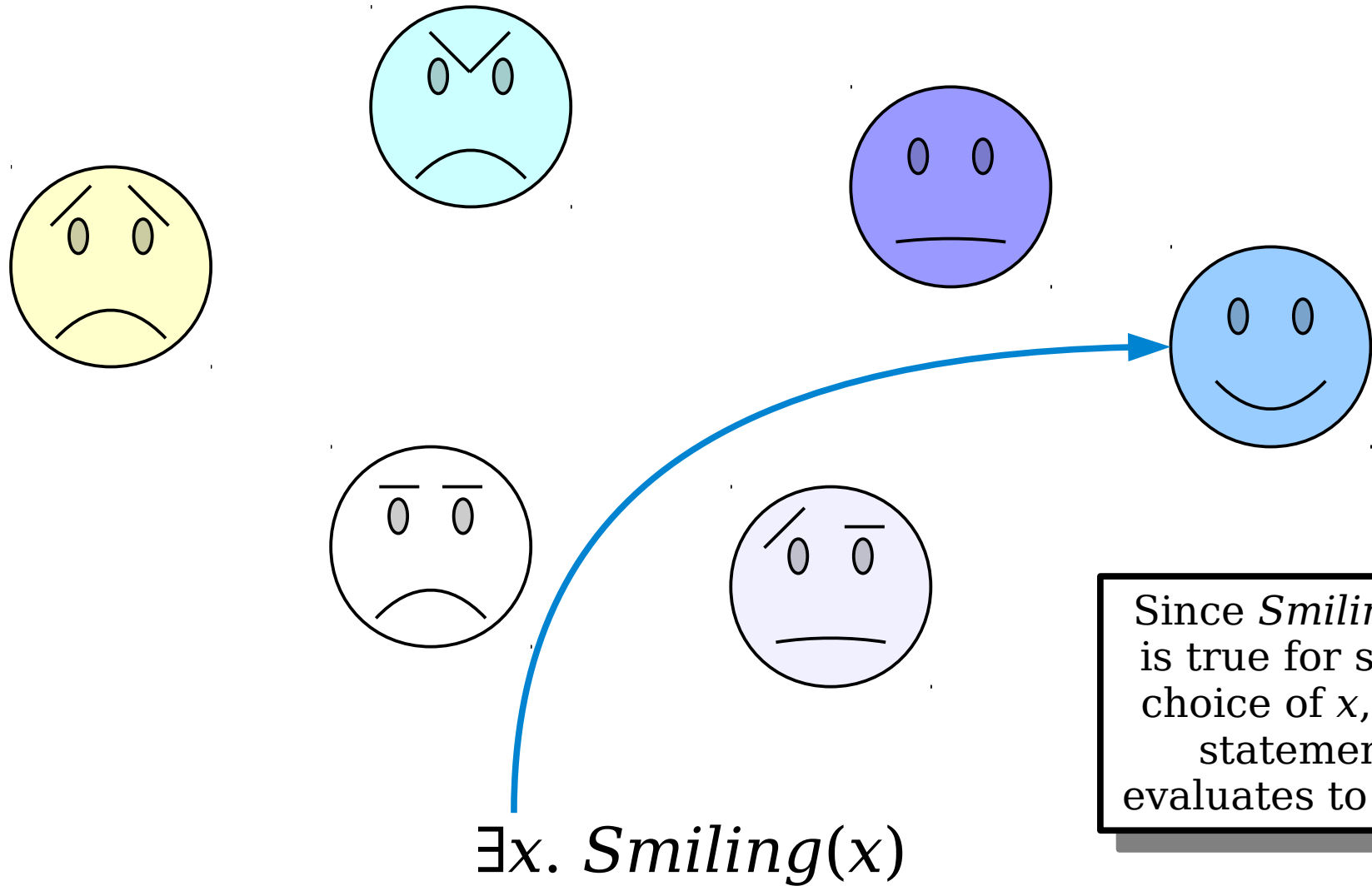
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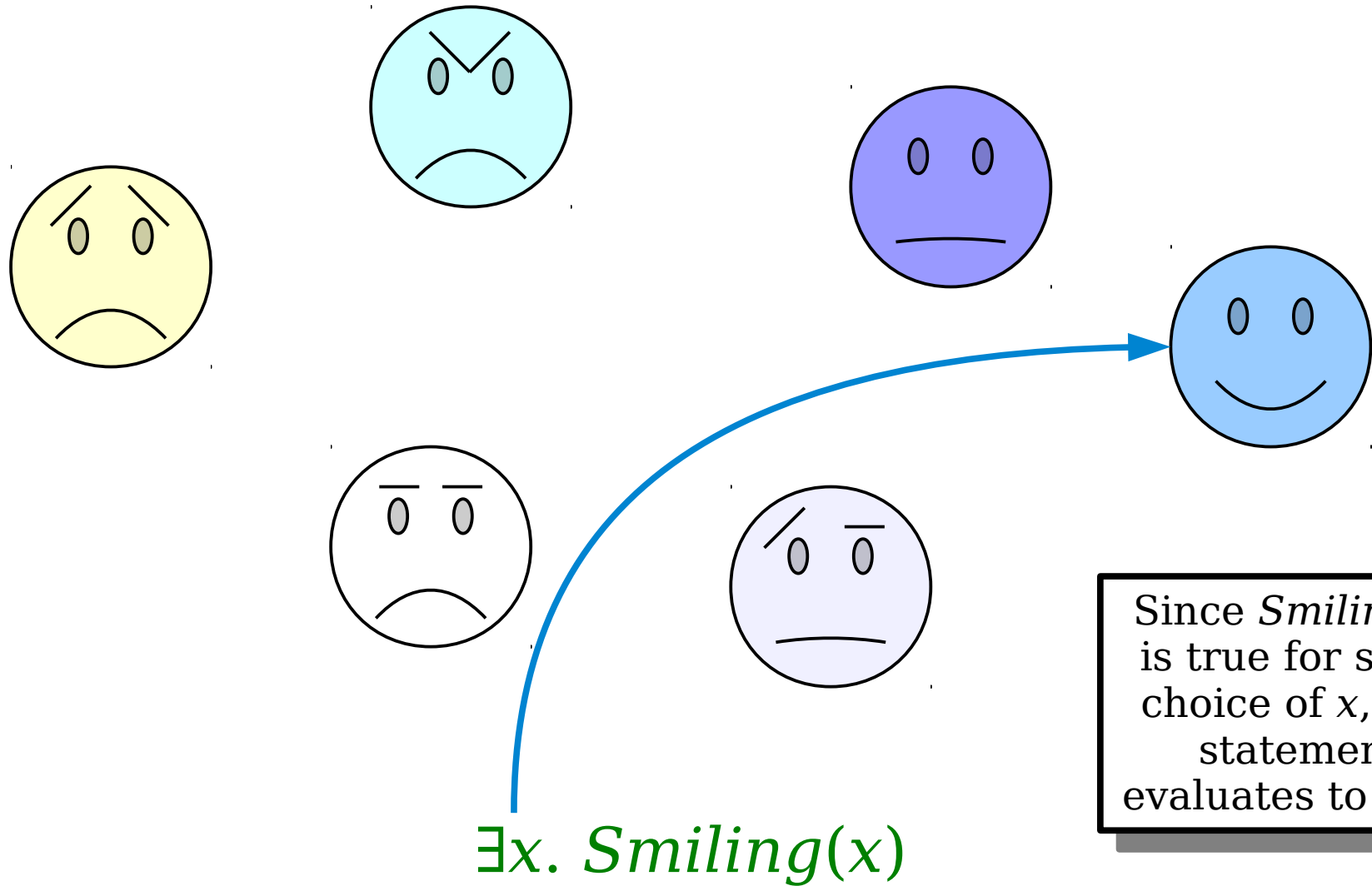
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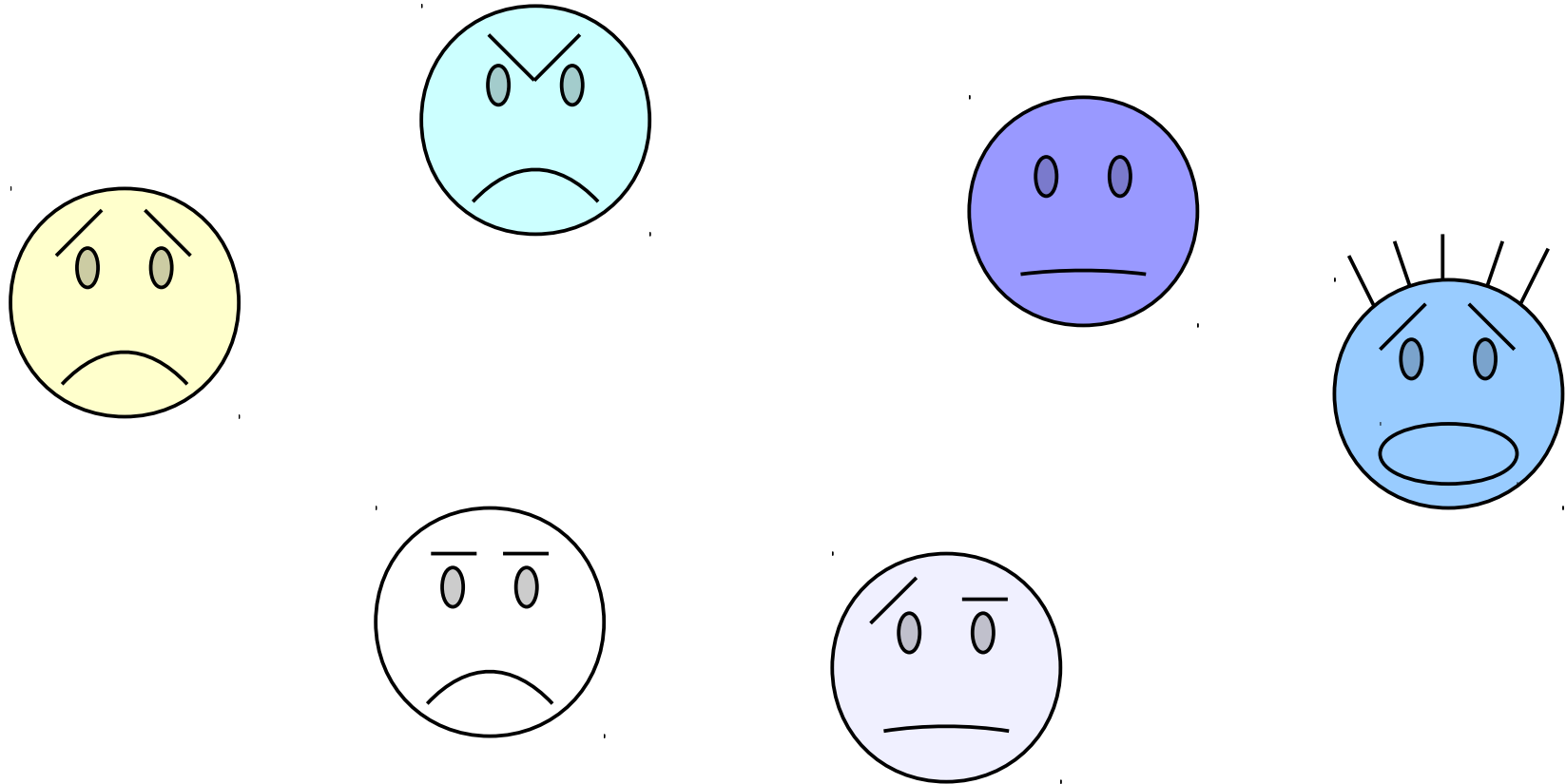


The Existential Quantifier



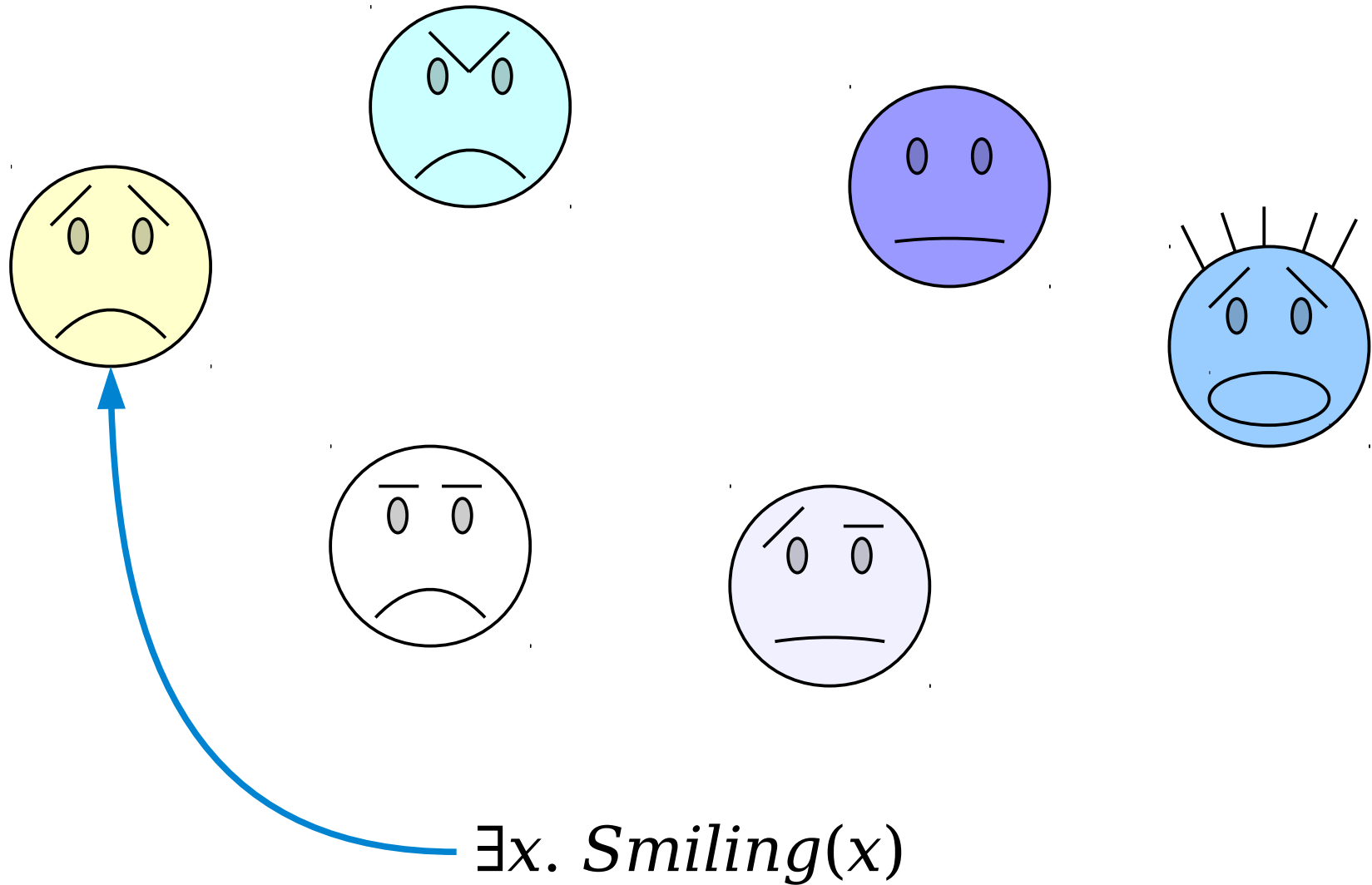
Since *Smiling(x)* is true for some choice of *x*, this statement evaluates to true.

The Existential Quantifier

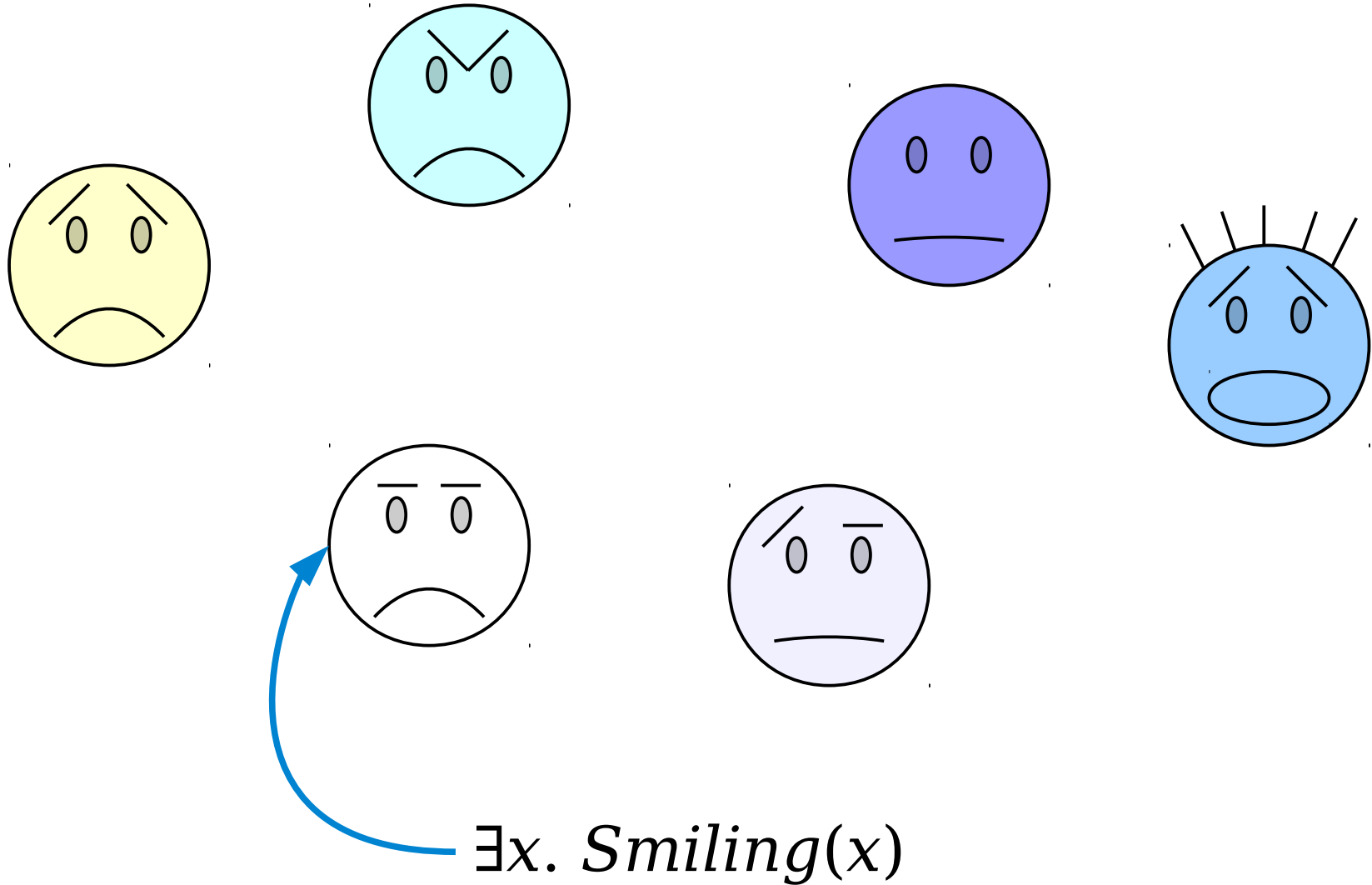


$\exists x. \textit{Smiling}(x)$

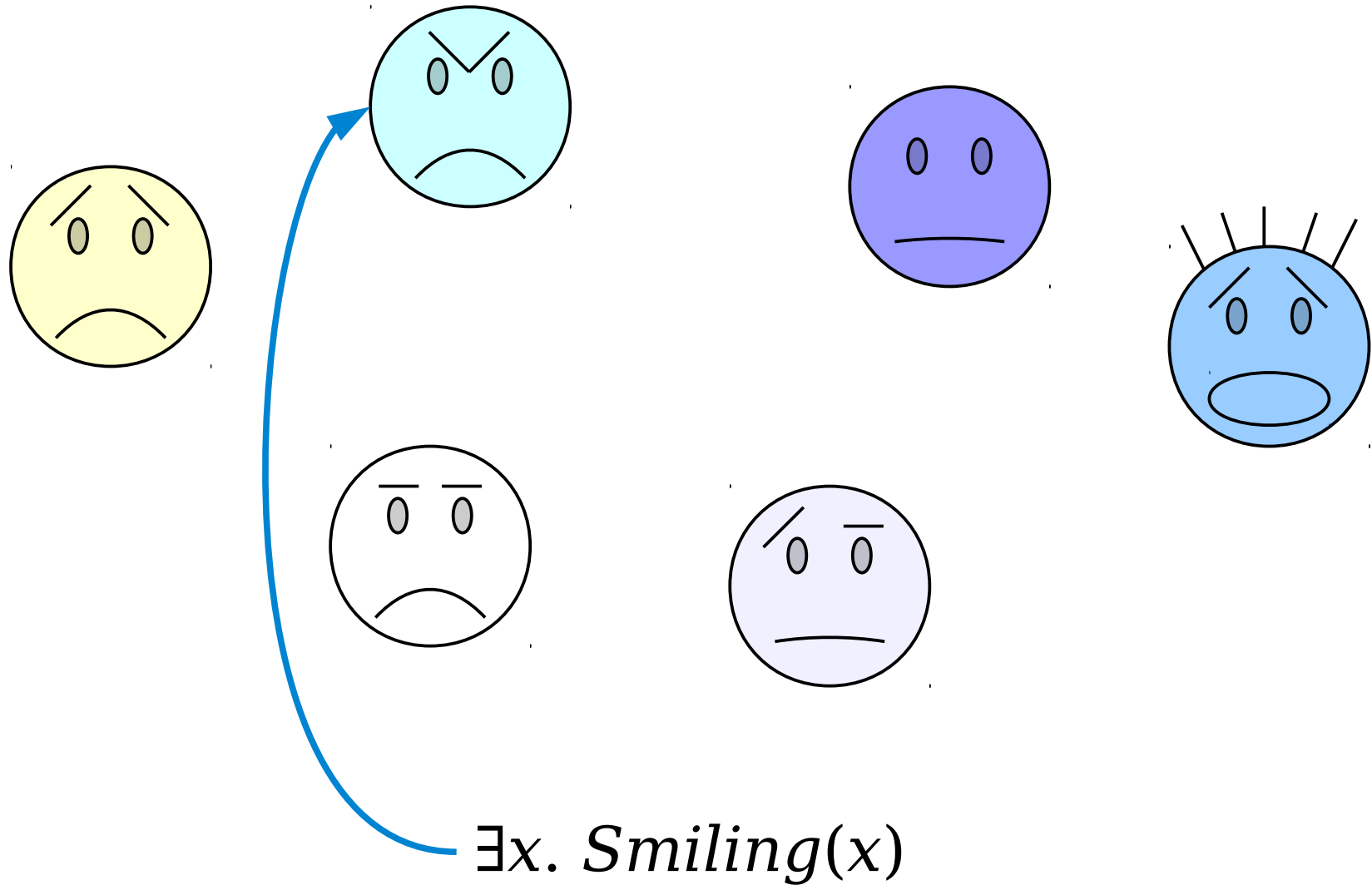
The Existential Quantifier



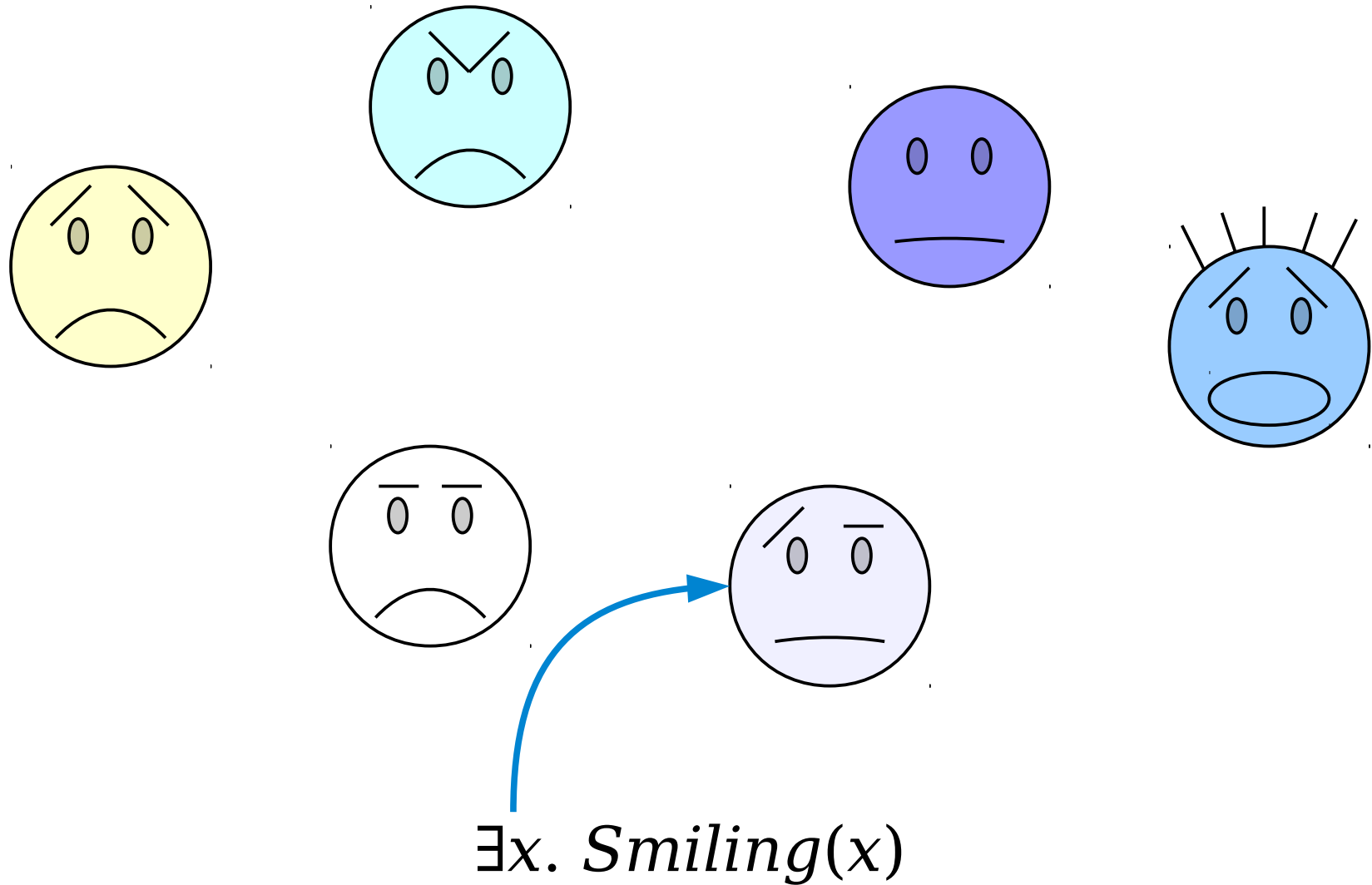
The Existential Quantifier



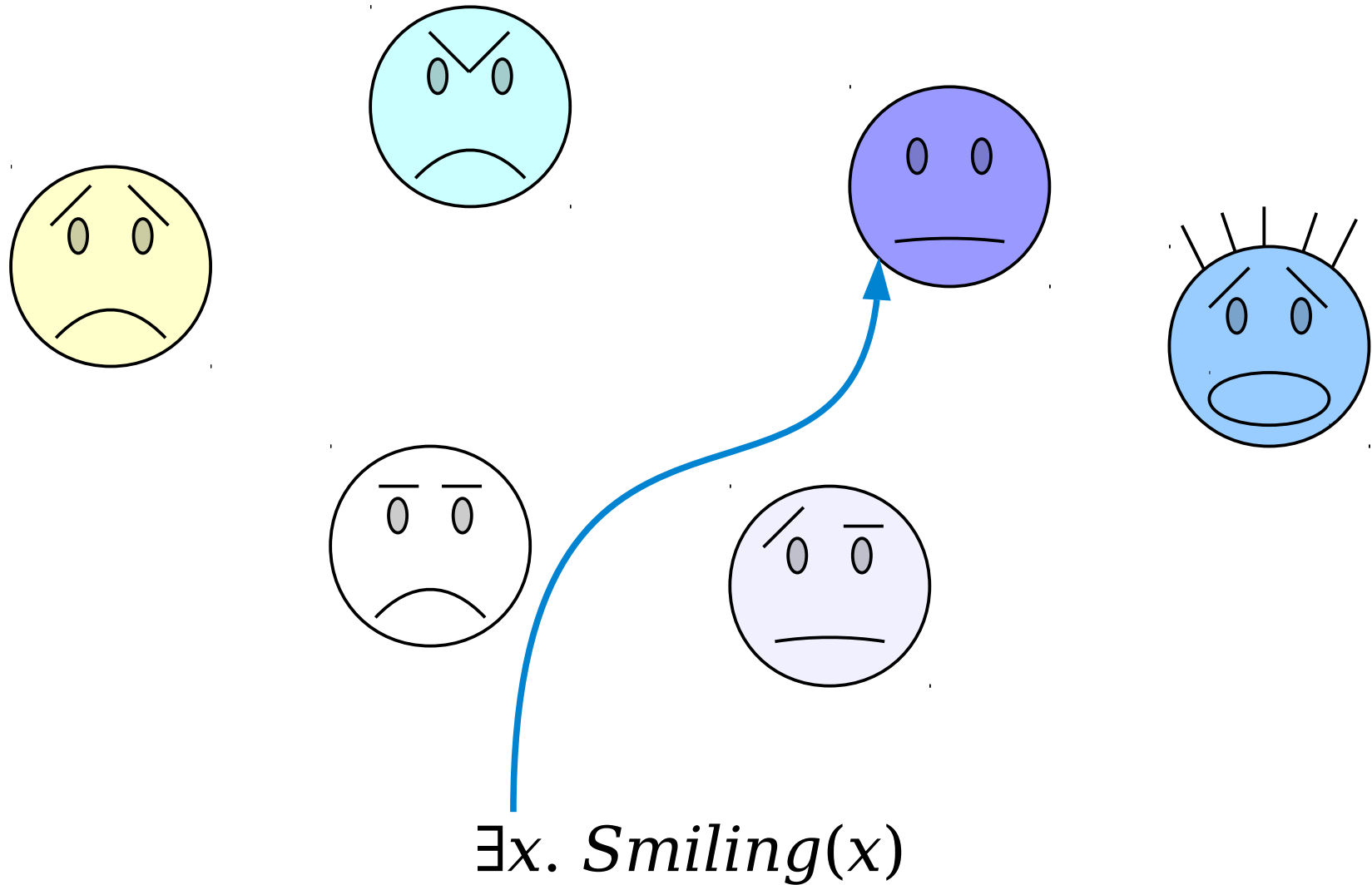
The Existential Quantifier



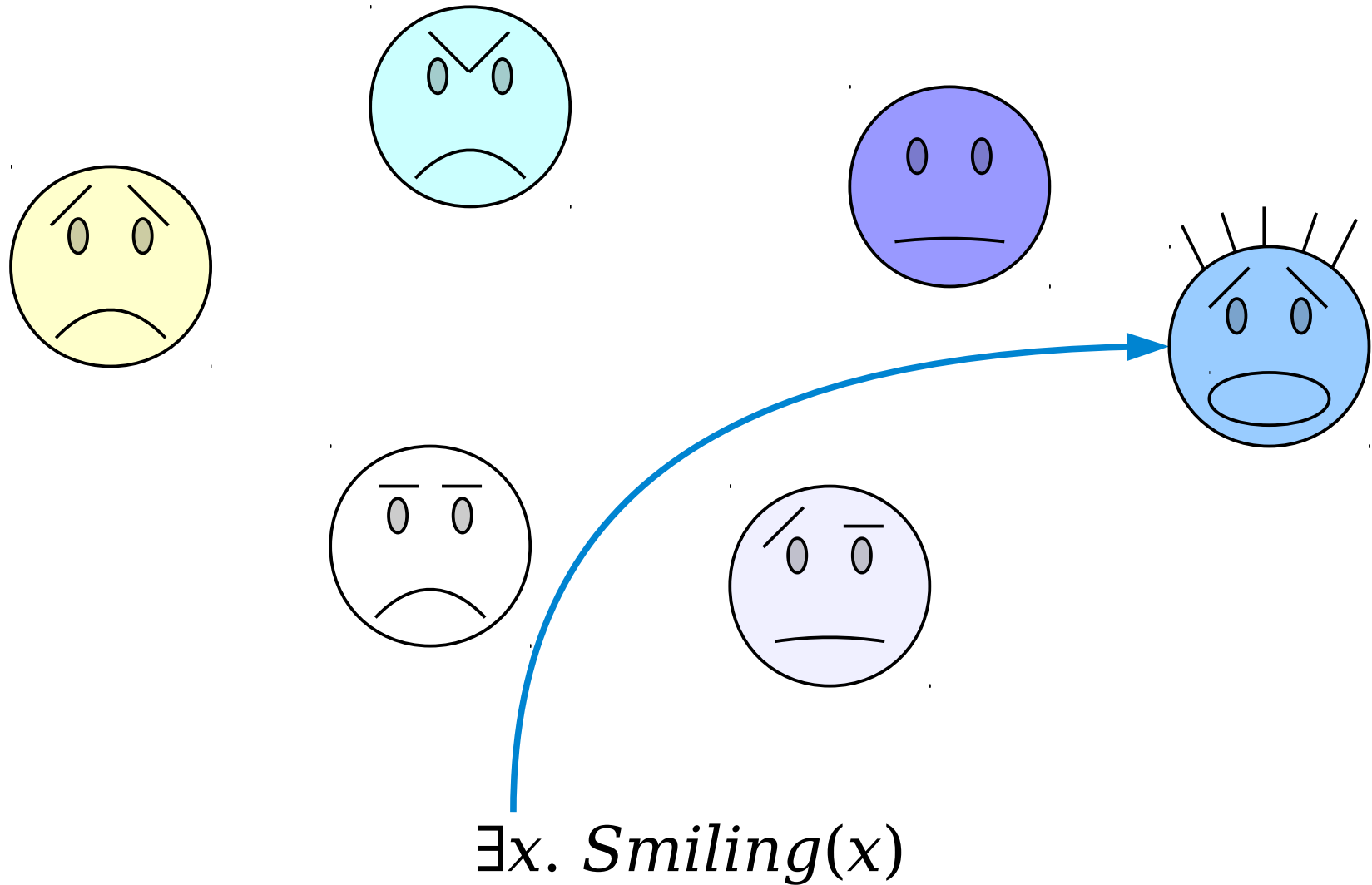
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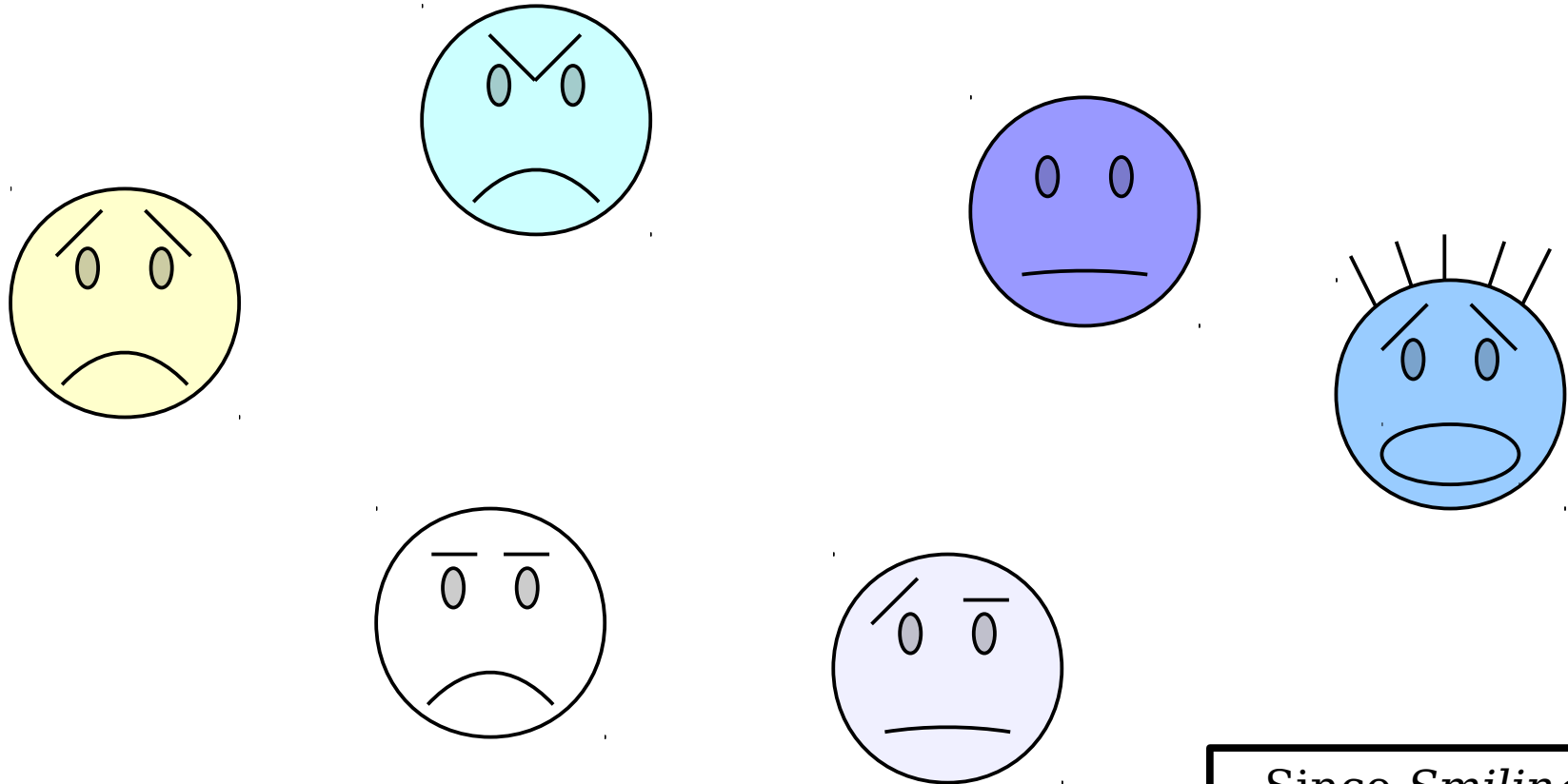
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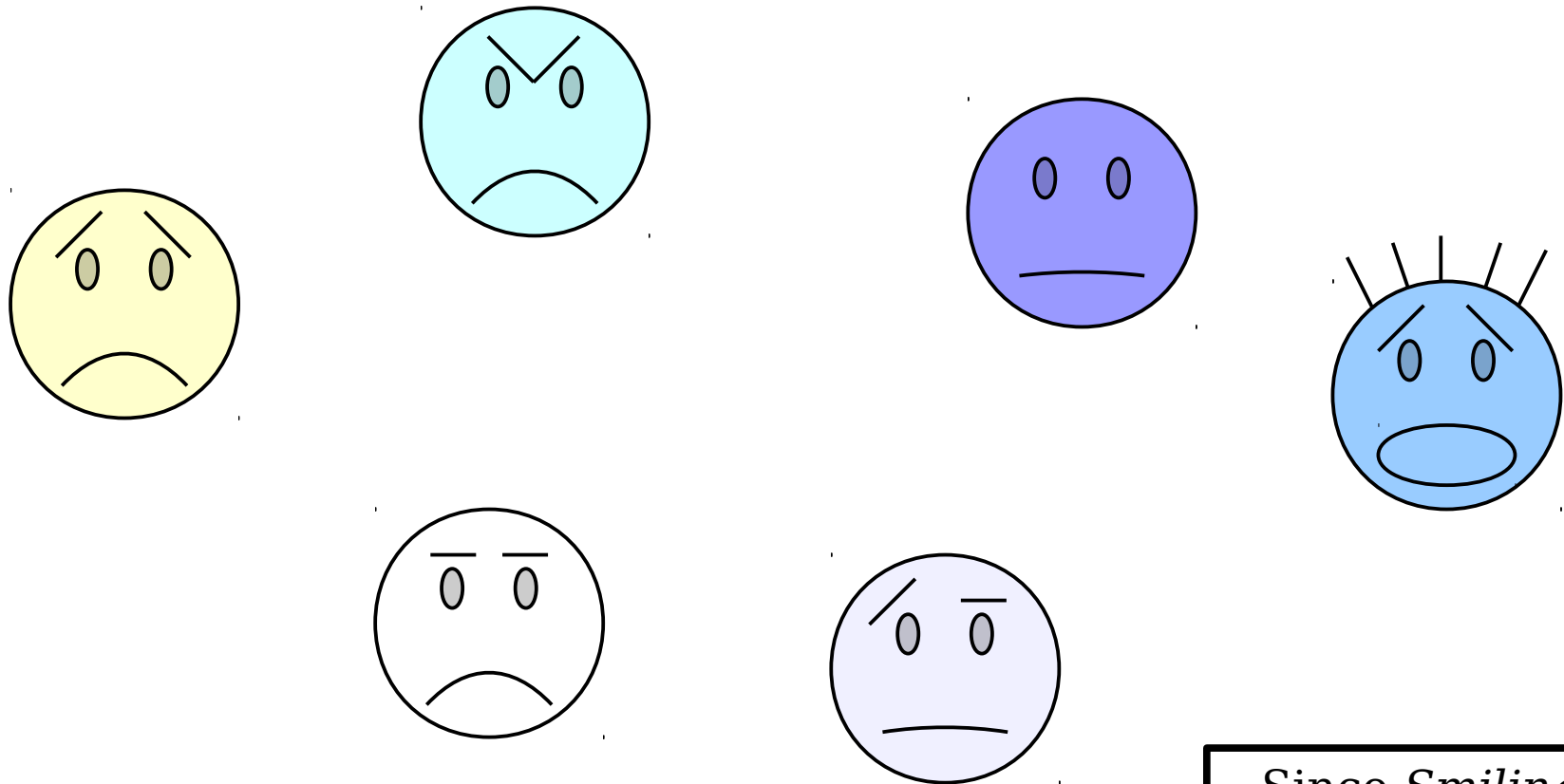
The Existential Quantifier



$\exists x. \textit{Smiling}(x)$

Since *Smiling*(x) is not true for any choice of x , this statement evaluates to false.

The Existential Quantifier

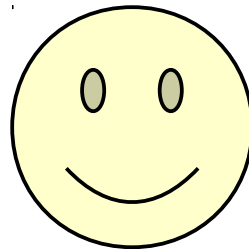
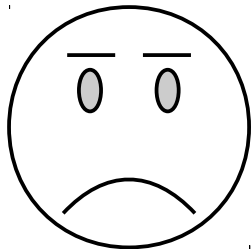
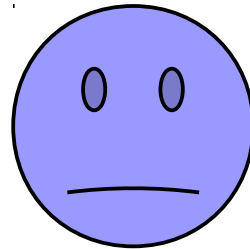
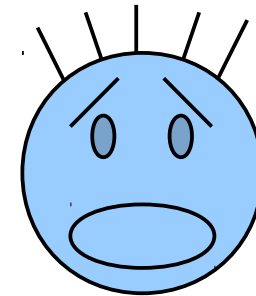
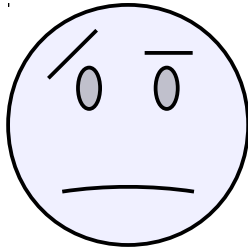


~~$\exists x. Smiling(x)$~~

Since $Smiling(x)$ is not true for any choice of x , this statement evaluates to false.

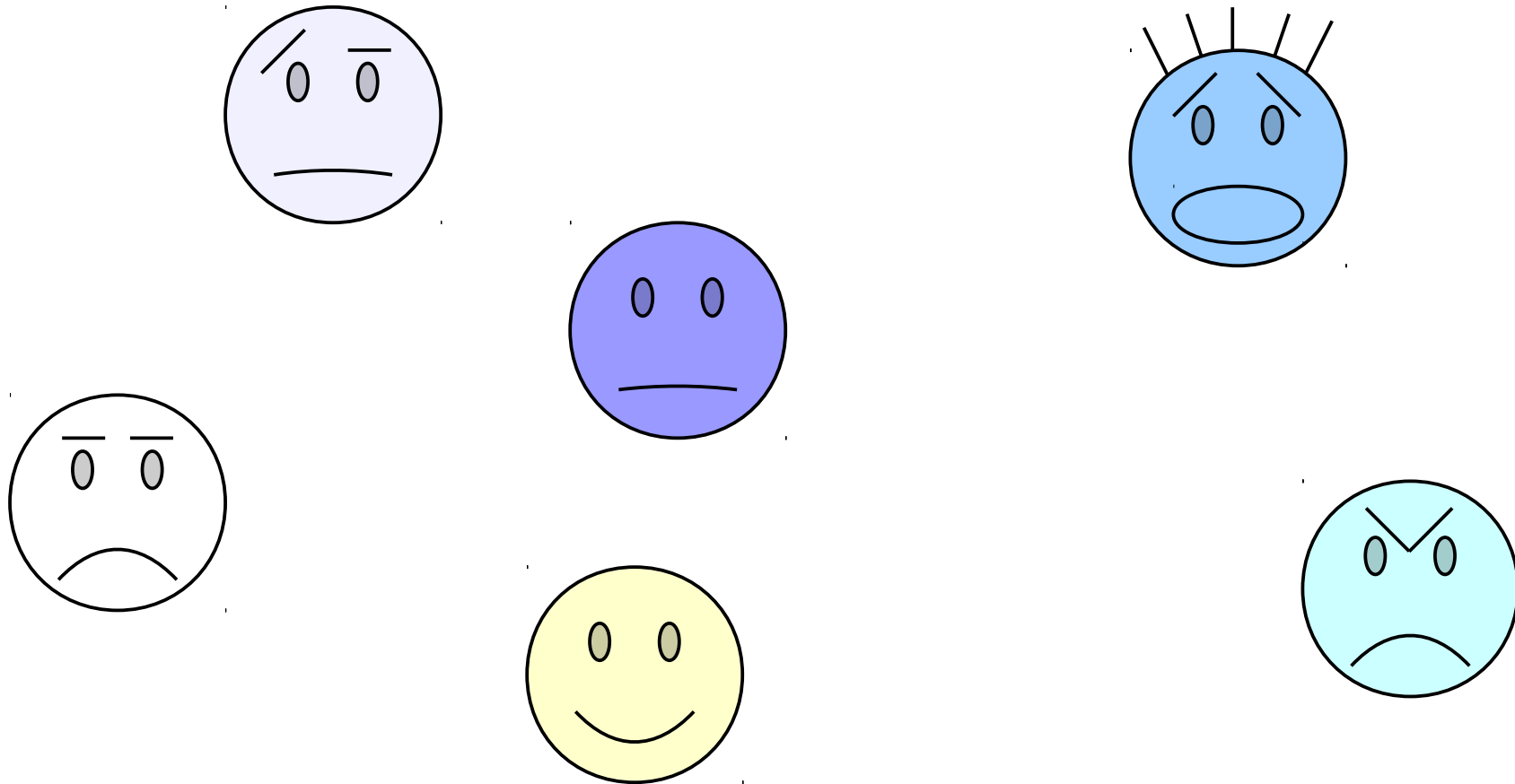
Question: In this world, is the first-order logic statement below true or false?

Respond at pollev.com/cs103



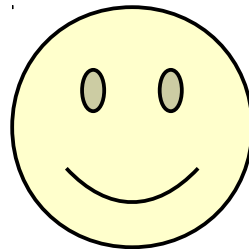
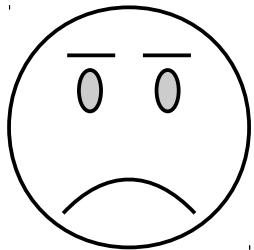
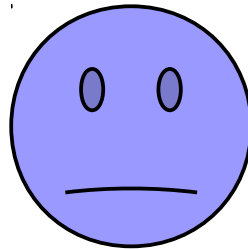
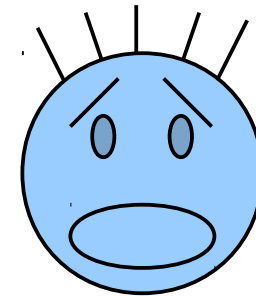
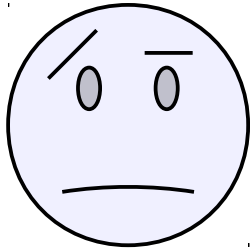
$(\exists x. Smiling(x)) \rightarrow (\exists y. WearingHat(y))$

The Existential Quantifier



$(\exists x. \textit{Smiling}(x)) \rightarrow (\exists y. \textit{WearingHat}(y))$

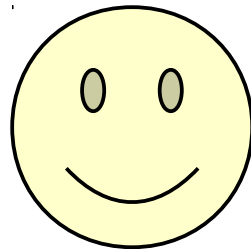
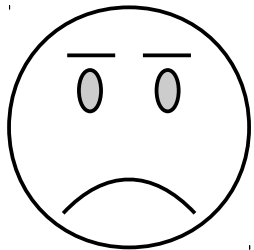
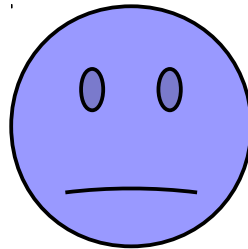
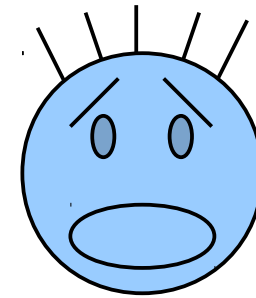
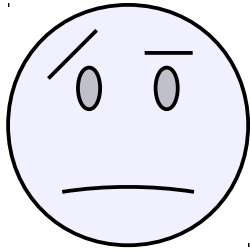
The Existential Quantifier



Is this part of the statement true or false?

$$(\exists x. \textit{Smiling}(x)) \rightarrow (\exists y. \textit{WearingHat}(y))$$

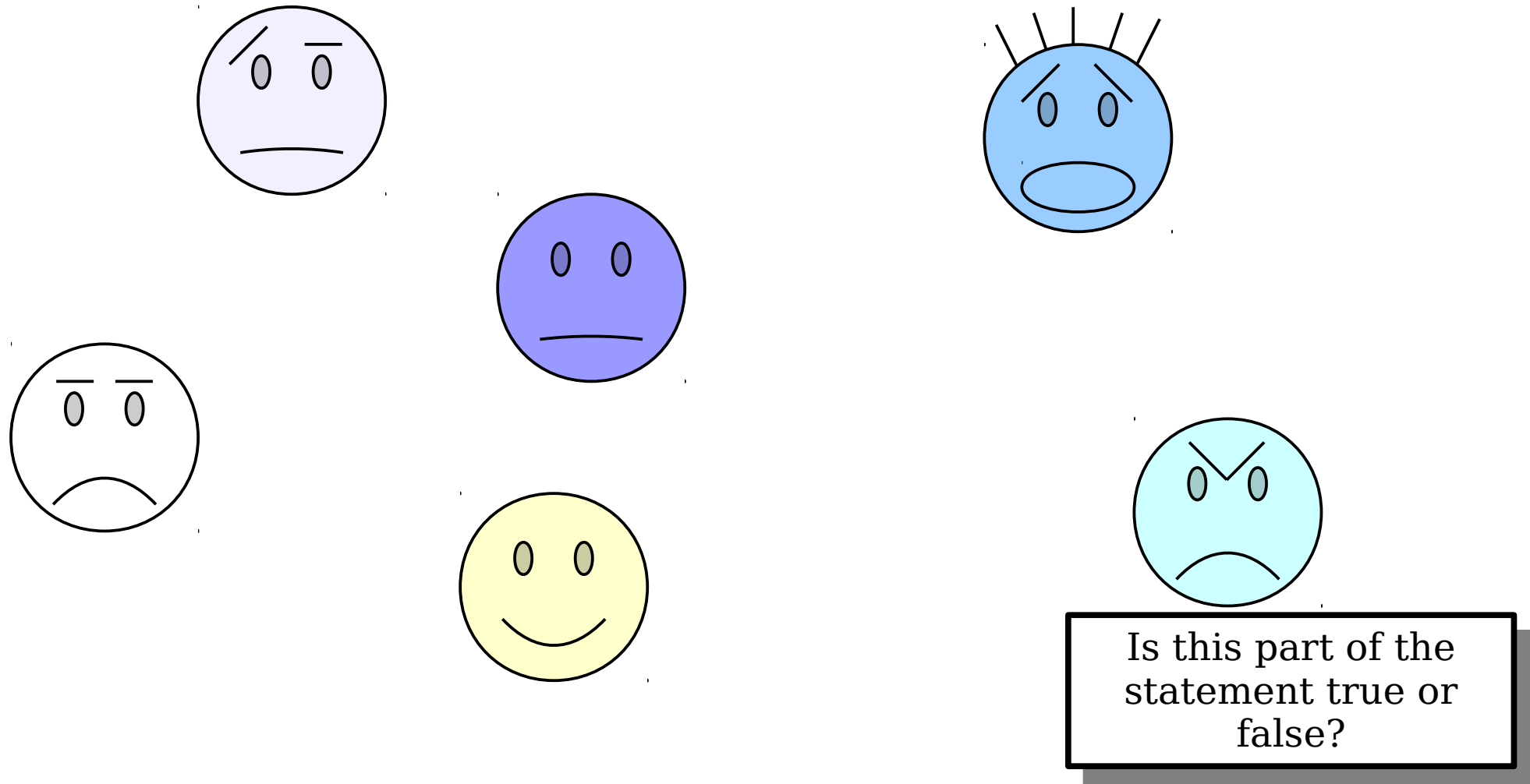
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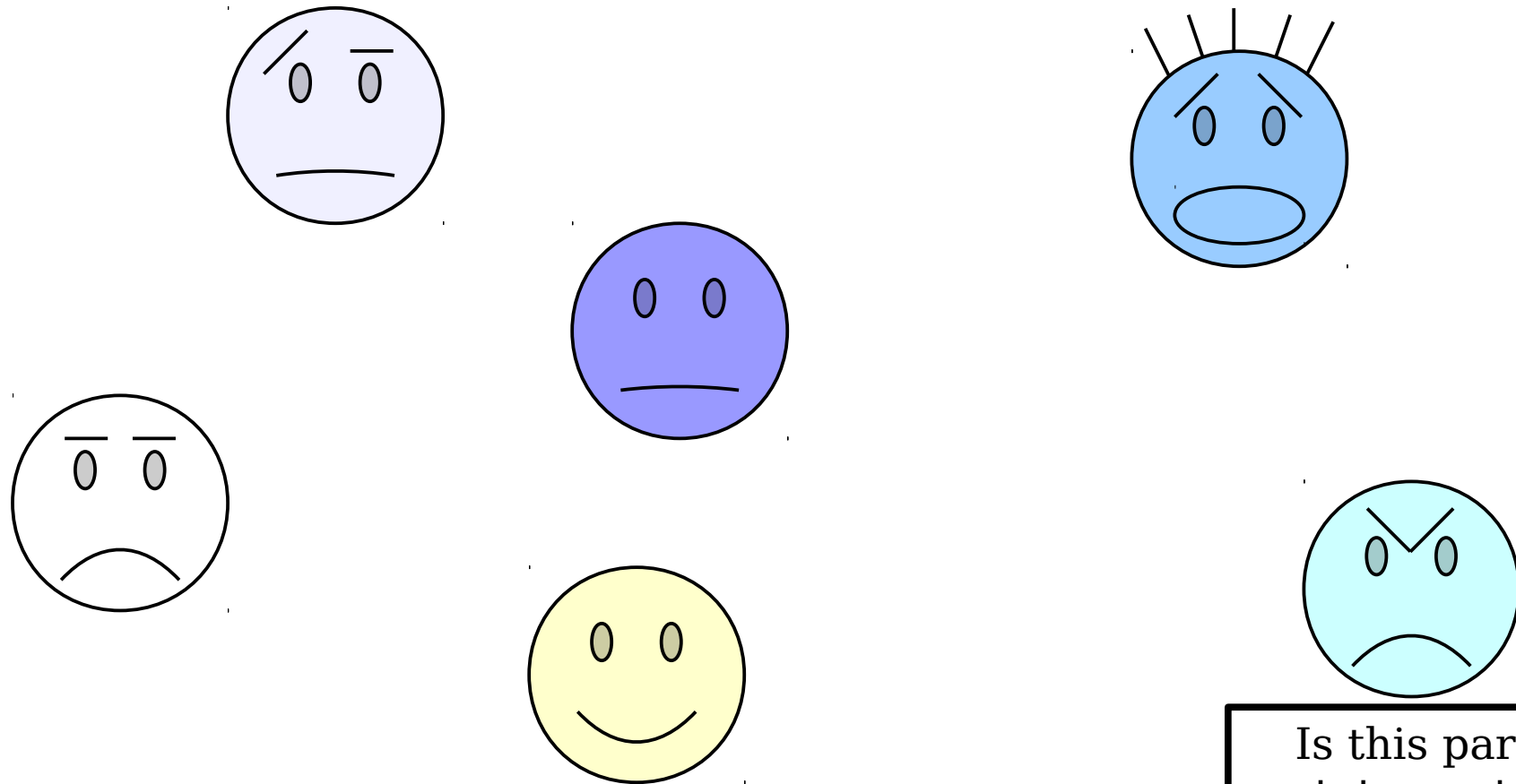
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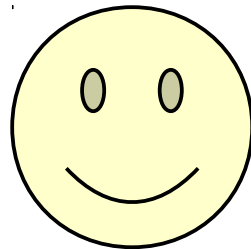
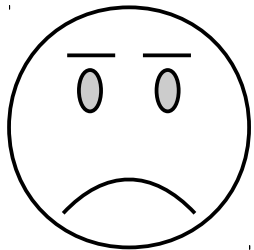
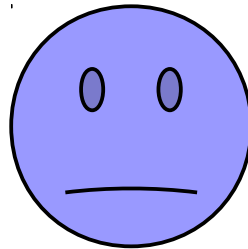
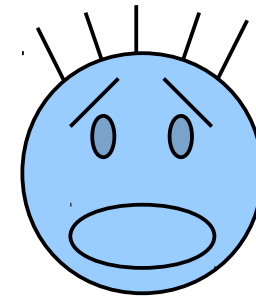
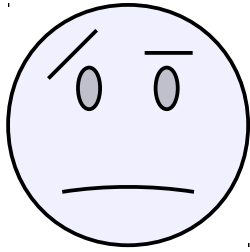
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Is this part of the statement true or false?

$(\exists x. \textit{Smiling}(x)) \rightarrow (\exists y. \textit{WearingHat}(y))$

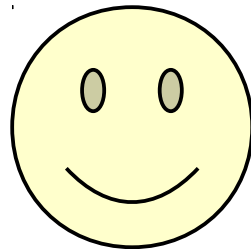
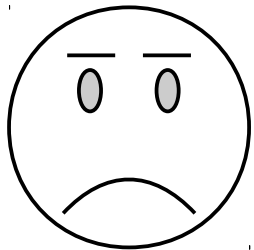
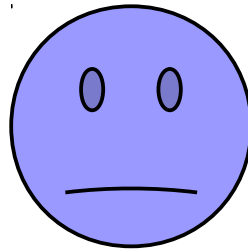
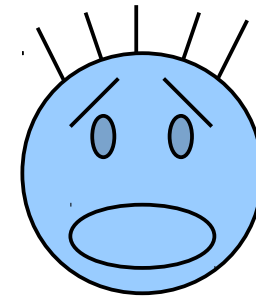
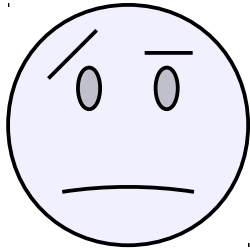
The Existential Quantifier



Is this overall
statement true or
false?

$$(\exists x. \textit{Smiling}(x)) \rightarrow (\exists y. \textit{WearingHat}(y))$$

The Existential Quantifier



Is this overall
statement true or
false?

~~$(\exists x. Smiling(x)) \rightarrow (\exists y. WearingHat(y))$~~

Fun with Edge Cases

$\exists x. \textit{Smiling}(x)$

Fun with Edge Cases

Existentially-quantified statements are false in an empty world, since nothing exists, period!

~~$\exists x. \textit{Smiling}(x)$~~

Some Technical Details

Variables and Quantifiers

- Each quantifier has two parts:
 - the variable that is introduced, and
 - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.

$(\exists x. \text{Loves}(\text{You}, x)) \wedge (\exists y. \text{Loves}(y, \text{You}))$

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The variable x
just lives
here.

The variable y
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The variable x
just lives
here.

A different variable,
also named x , just
lives here.

Operator Precedence (Again)

- When writing out a formula in first-order logic, quantifiers have precedence just below \neg .
- The statement

$$\exists x. P(x) \wedge R(x) \wedge Q(x)$$

is parsed like this:

$$\triangle (\exists x. P(x)) \wedge (R(x) \wedge Q(x)) \triangle$$

- This is syntactically invalid because the variable x is out of scope in the back half of the formula.
- To ensure that x is properly quantified, explicitly put parentheses around the region you want to quantify:

$$\exists x. (P(x) \wedge R(x) \wedge Q(x))$$

“For any natural number n ,
 n is even if and only if n^2 is even”

“For any natural number n ,
 n is even if and only if n^2 is even”

$\forall n. (n \in \mathbb{N} \rightarrow (Even(n) \leftrightarrow Even(n^2)))$

“For any natural number n ,
 n is even if and only if n^2 is even”

$\forall n. (n \in \mathbb{N} \rightarrow (Even(n) \leftrightarrow Even(n^2)))$

\forall is the **universal quantifier**
and says “for any choice of
 n , the following is true.”

The Universal Quantifier

- A statement of the form

$\forall x.$ *some-formula*

is true if, for every choice of x , the statement ***some-formula*** is true when x is plugged into it.

- Examples:

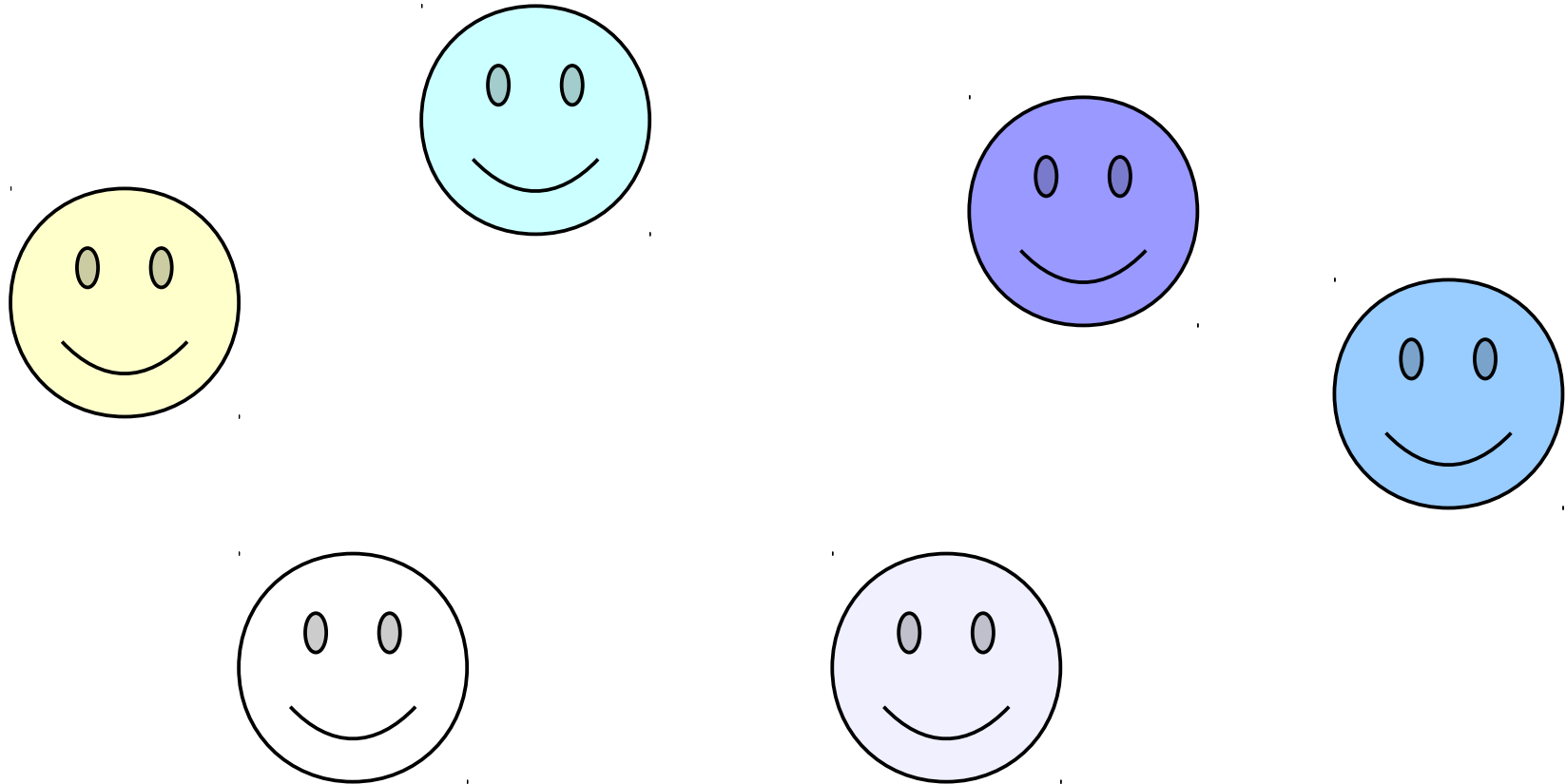
$\forall p. (Puppy(p) \rightarrow Cute(p))$

$\forall a. (EatsPlants(a) \vee EatsAnimals(a))$

$Tallest(SultanKösen) \rightarrow$

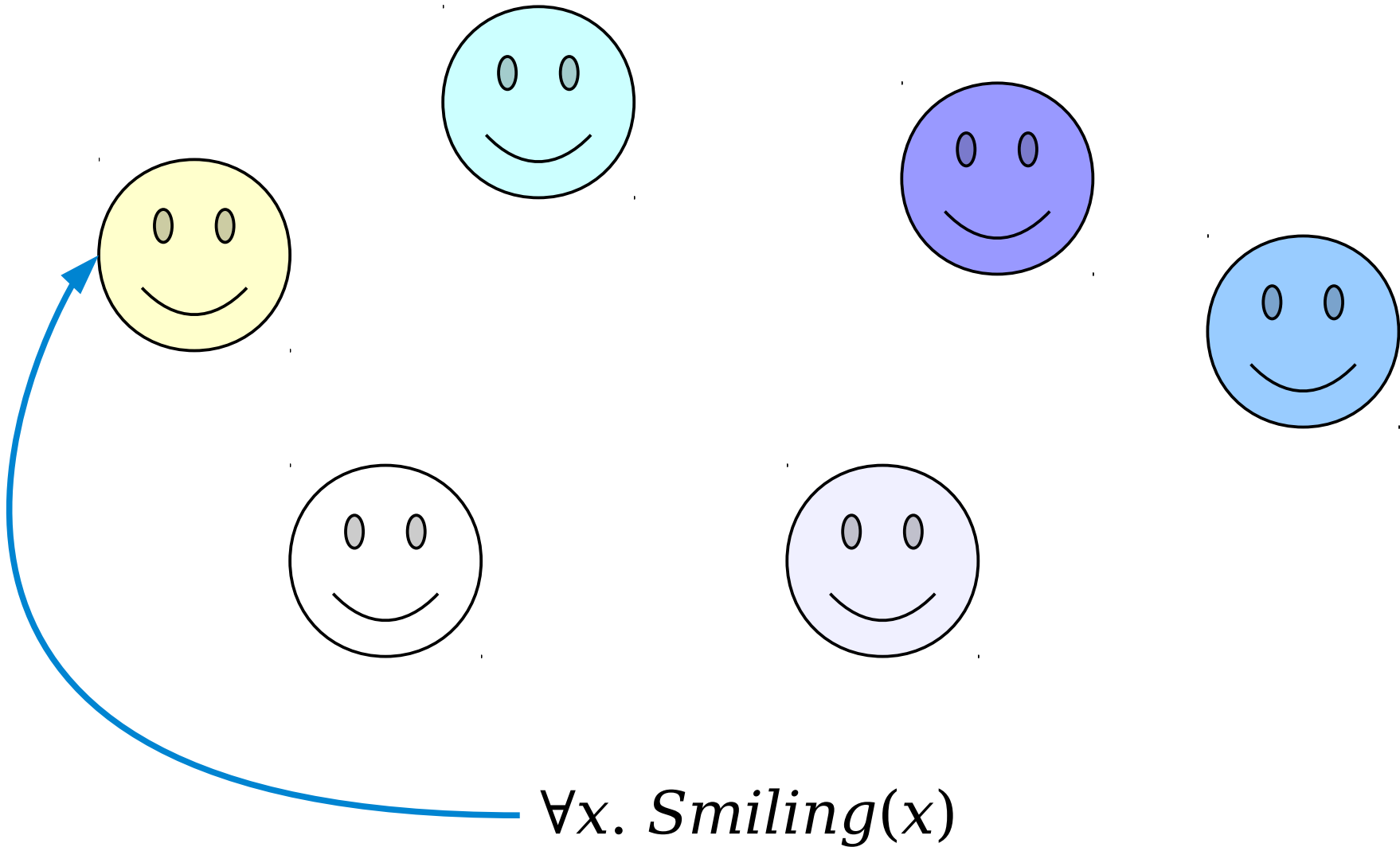
$\forall x. (SultanKösen \neq x \rightarrow ShorterThan(x, SultanKösen))$

The Universal Quantifier

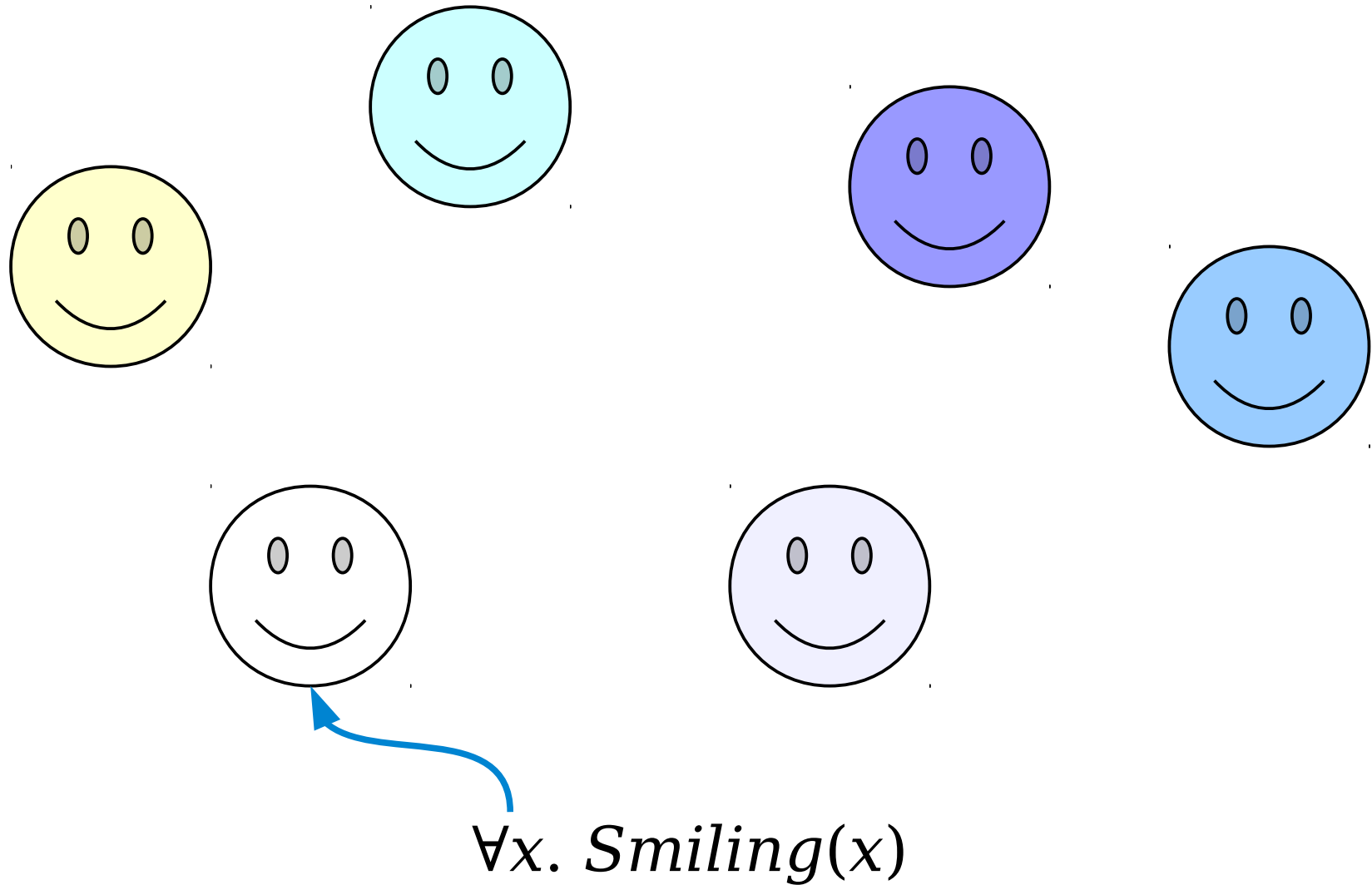


$\forall x. \textit{Smiling}(x)$

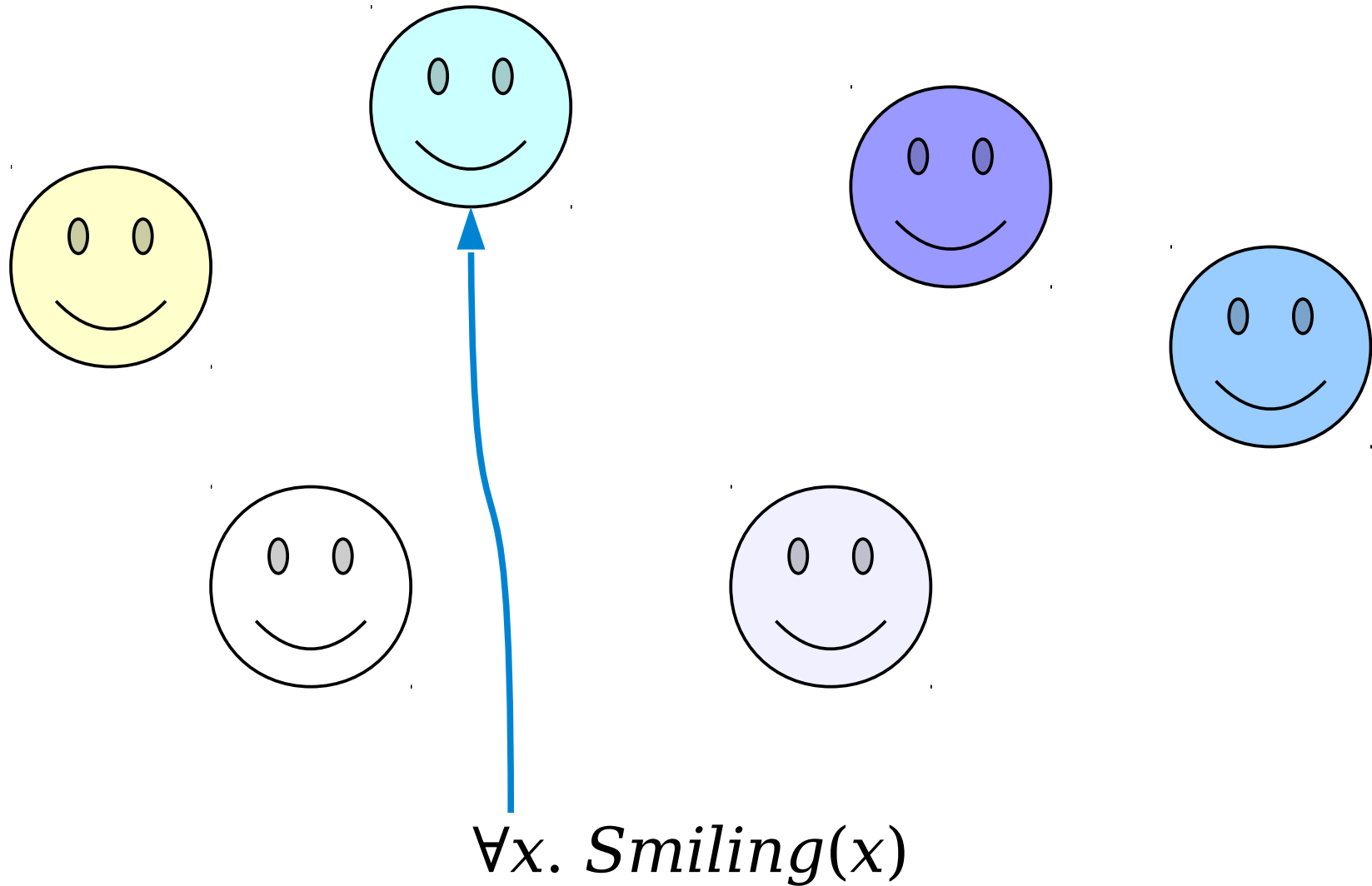
The Universal Quantifier



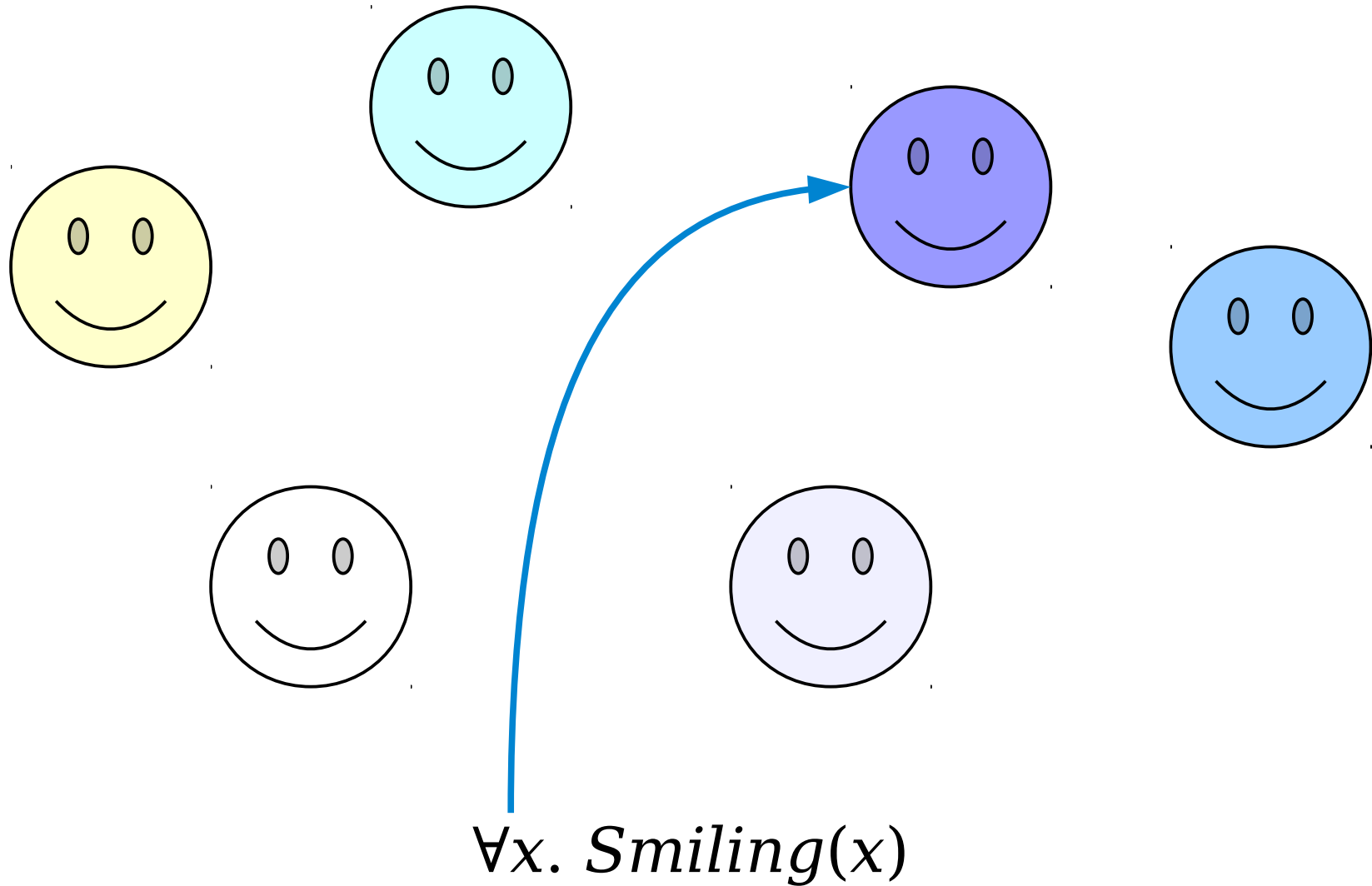
The Universal Quantifier



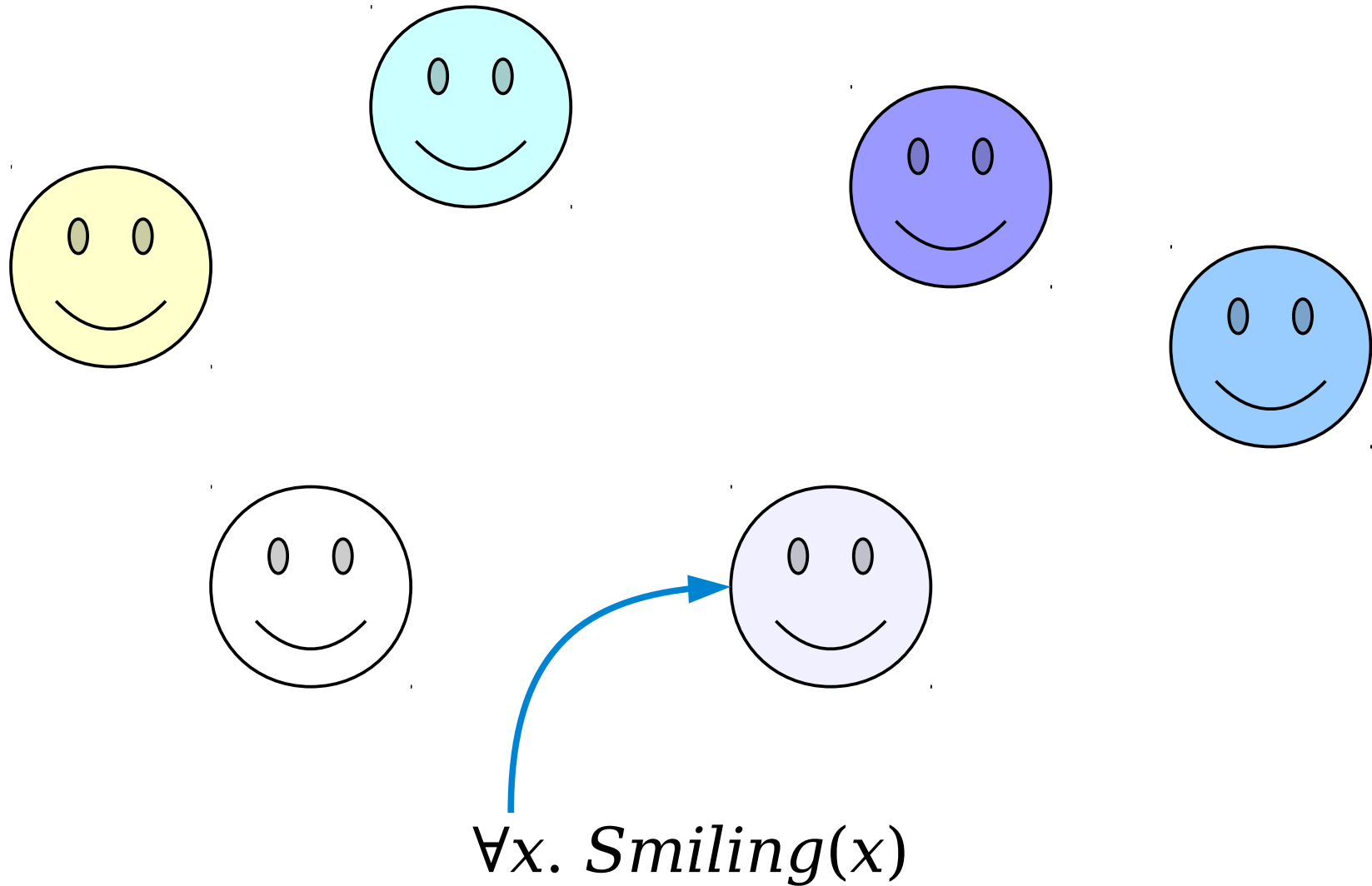
The Universal Quantifier



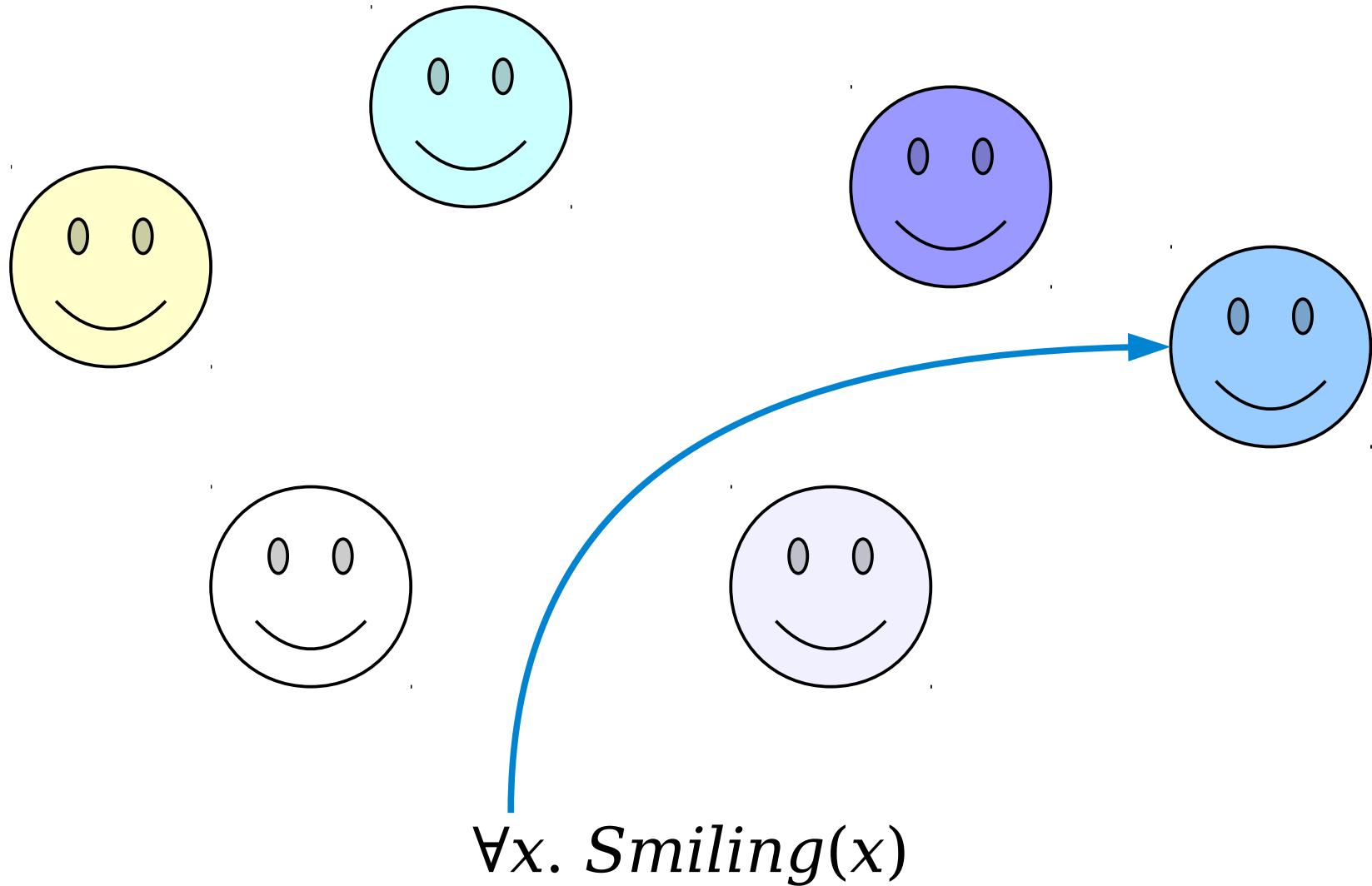
The Universal Quantifier



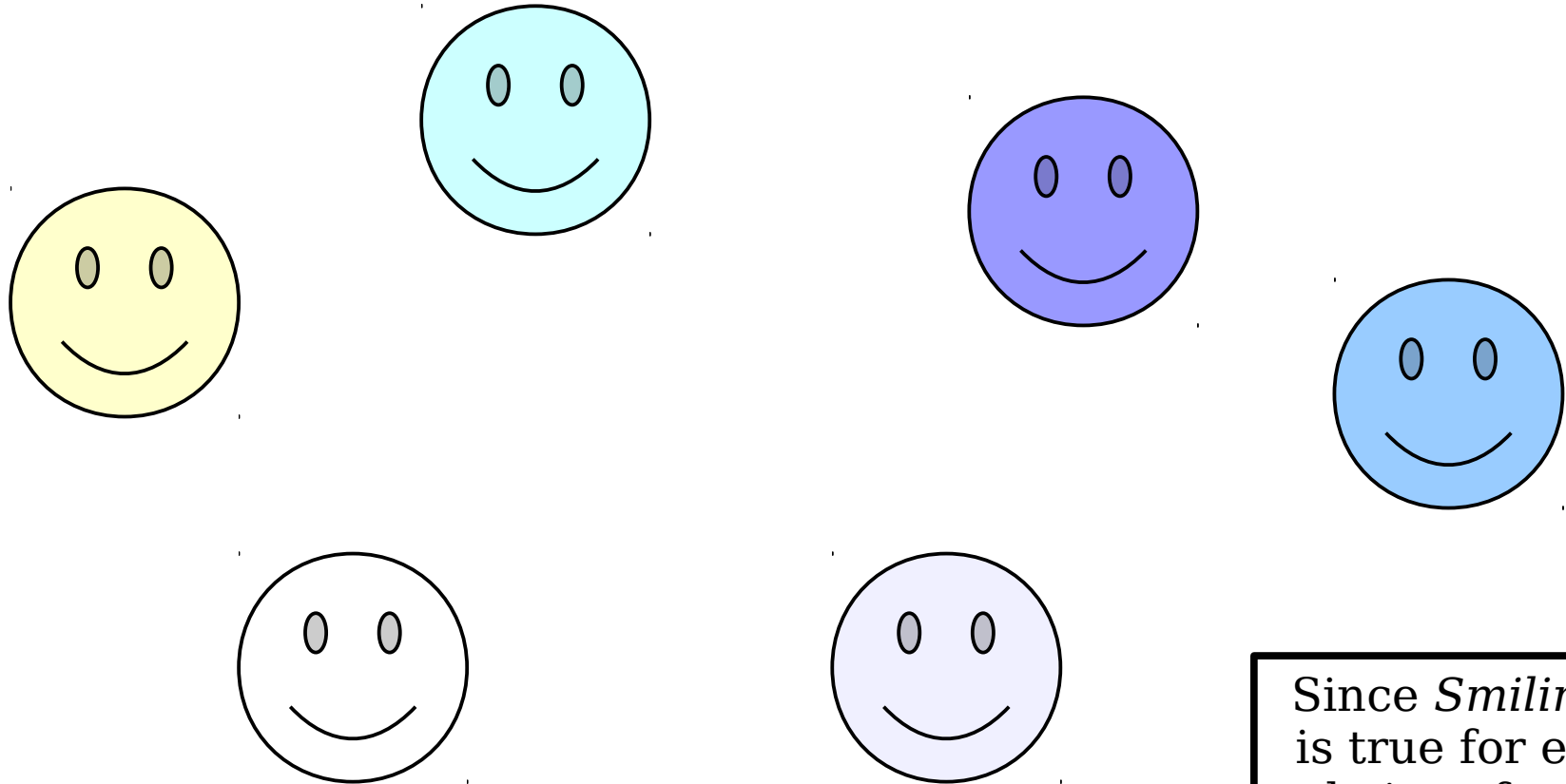
The Universal Quantifier



The Universal Quantifier



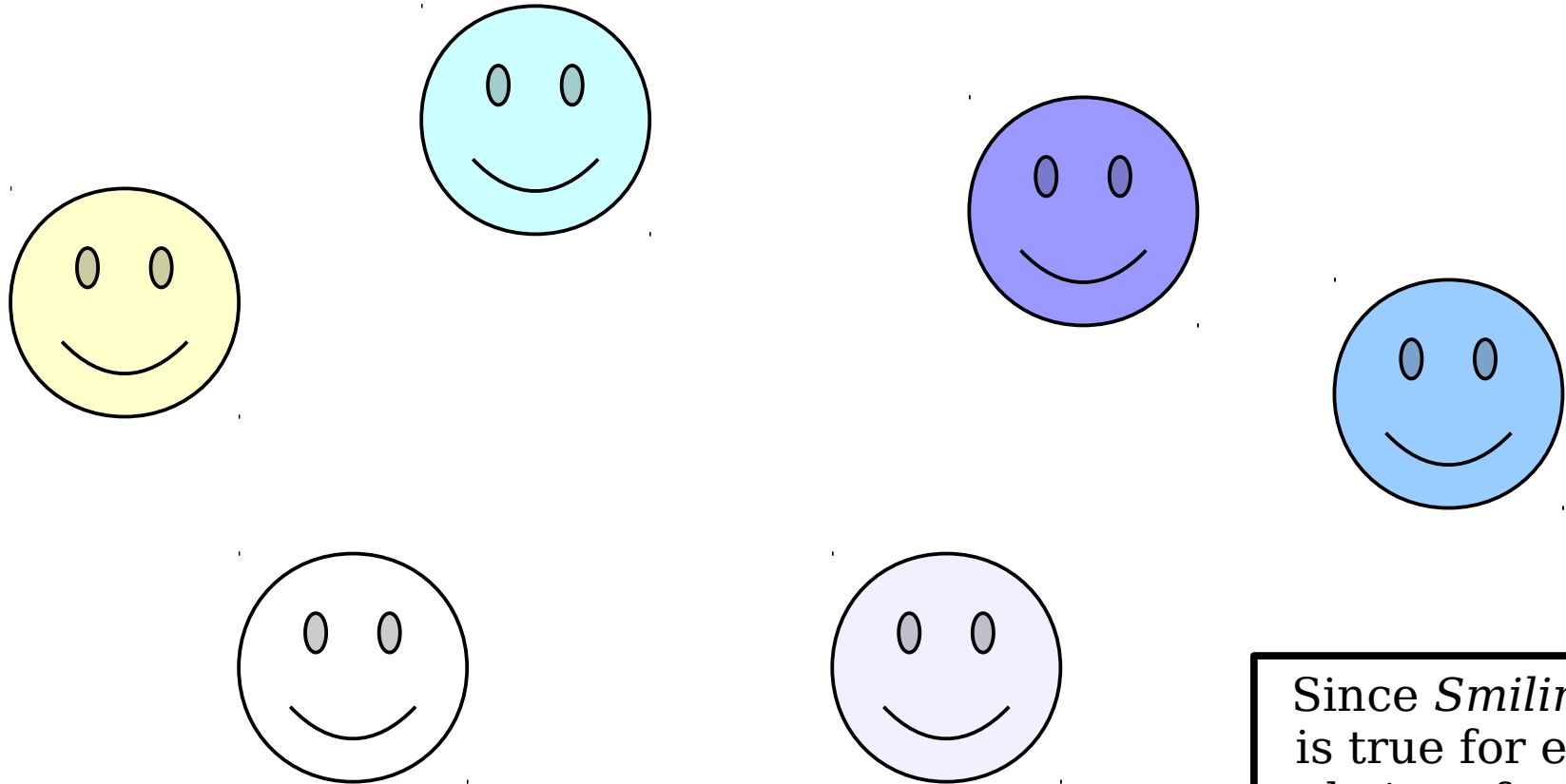
The Universal Quantifier



$\forall x. \textit{Smiling}(x)$

Since *Smiling*(x)
is true for every
choice of x , this
statement
evaluates to true.

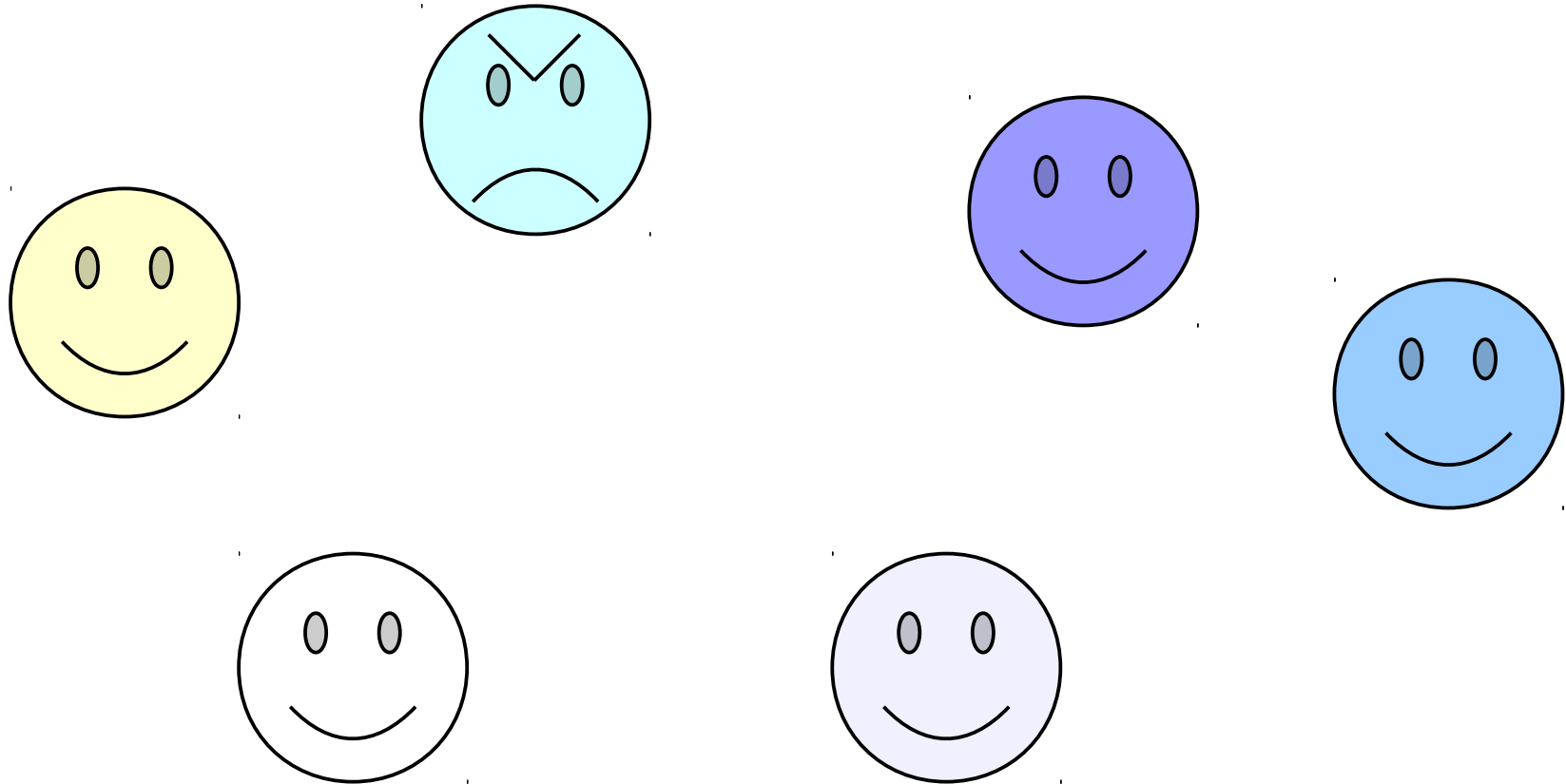
The Universal Quantifier



$\forall x. \textit{Smiling}(x)$

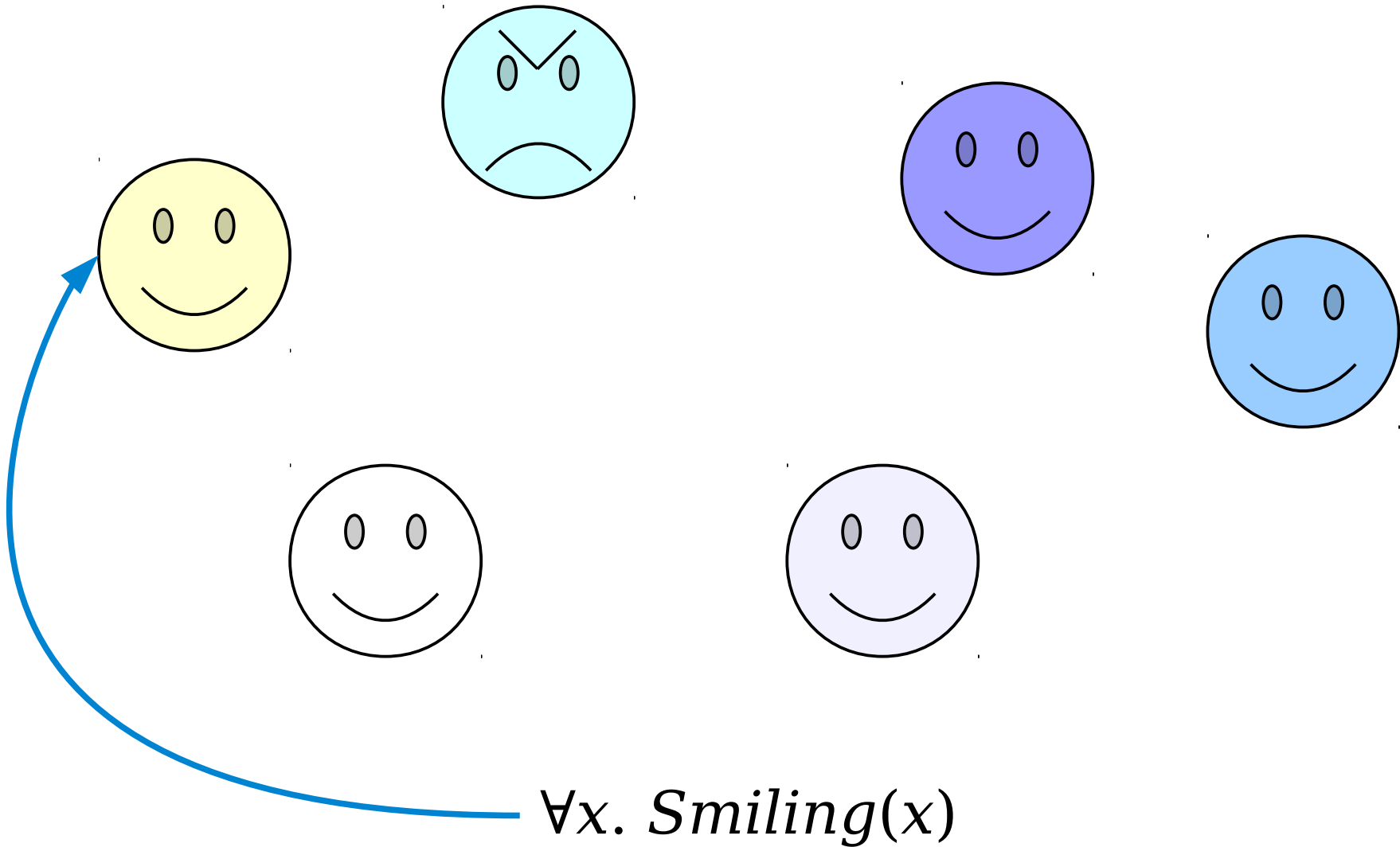
Since *Smiling*(*x*)
is true for every
choice of *x*, this
statement
evaluates to true.

The Universal Quantifier

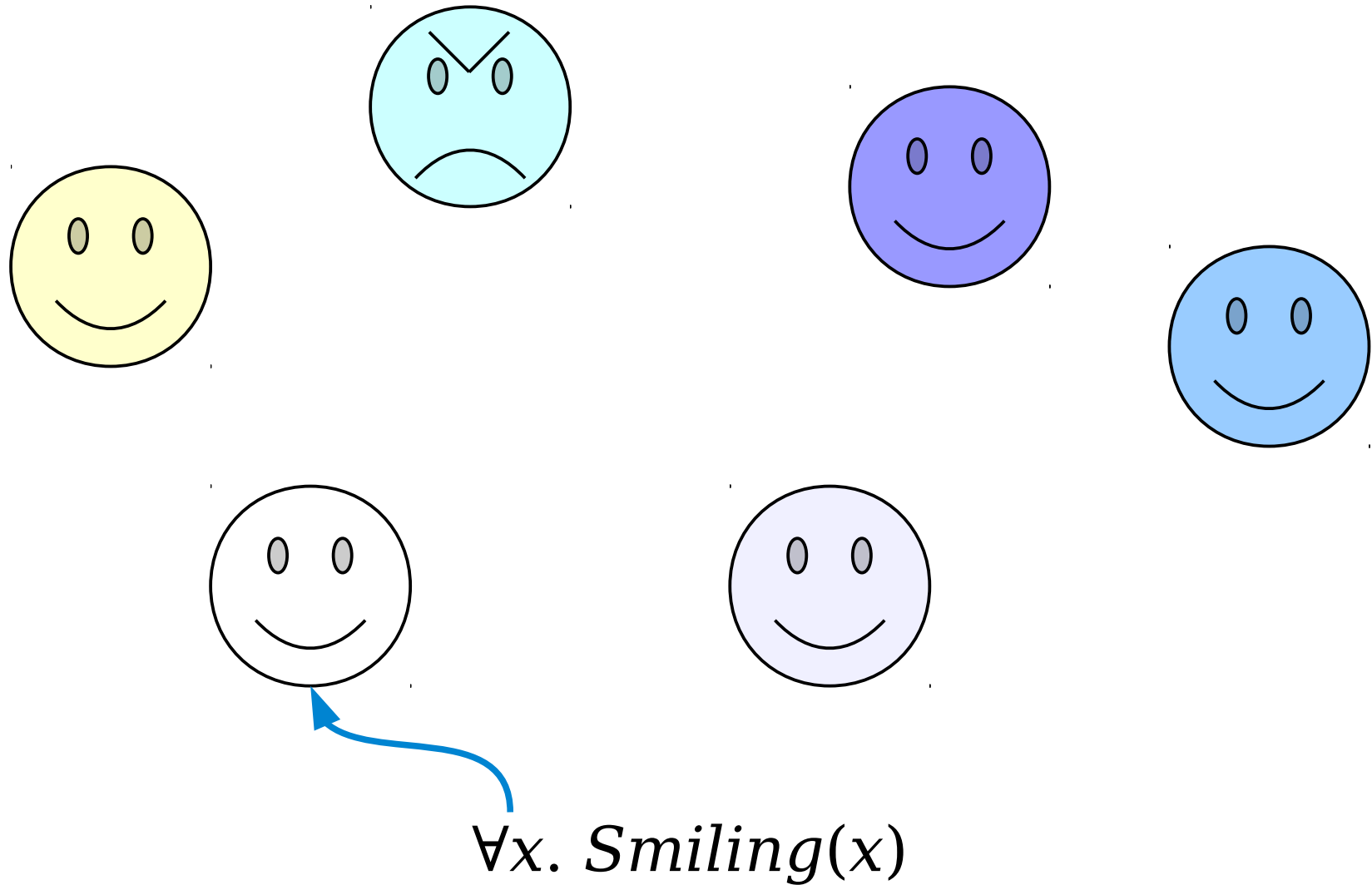


$\forall x. \textit{Smiling}(x)$

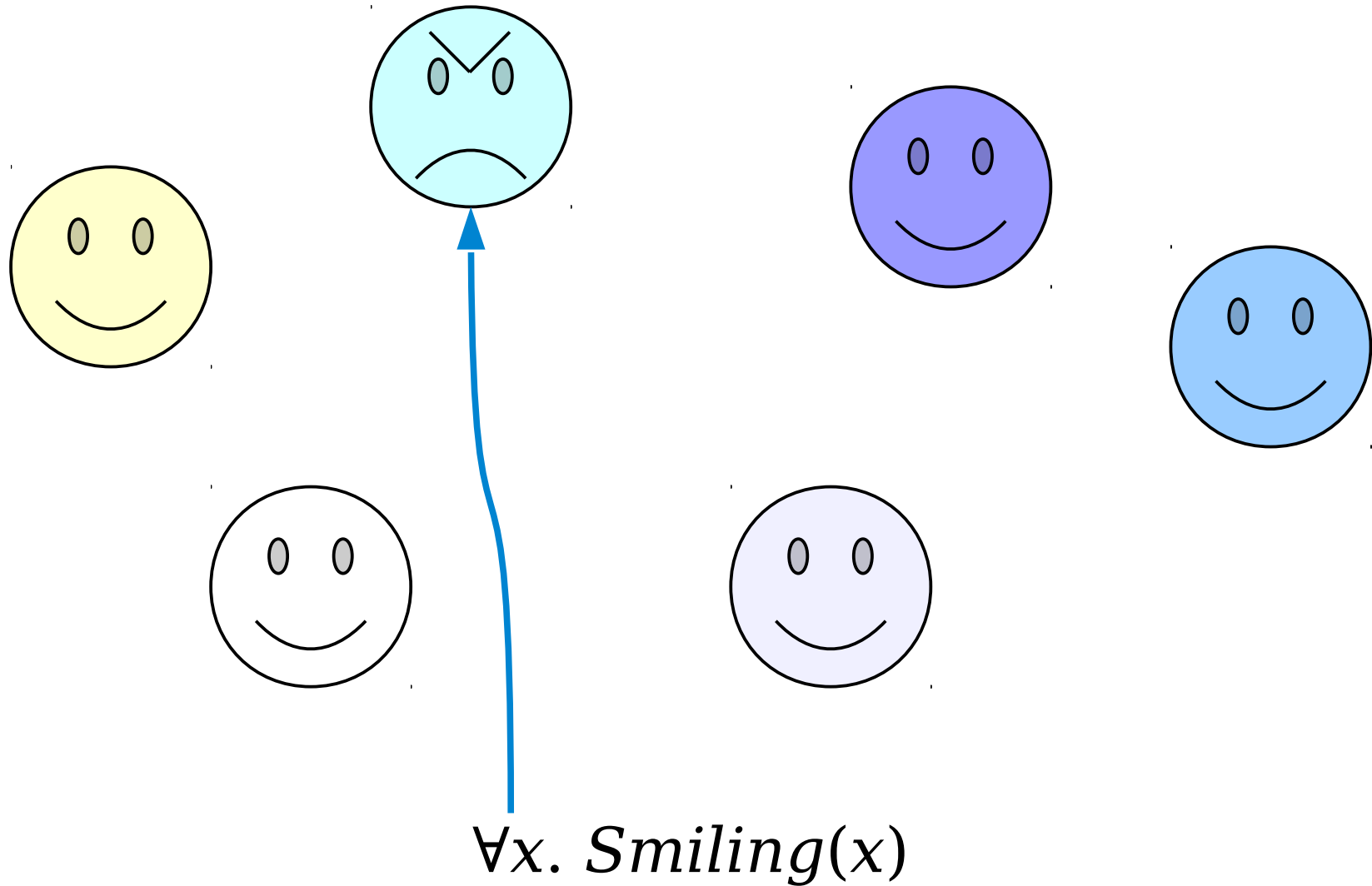
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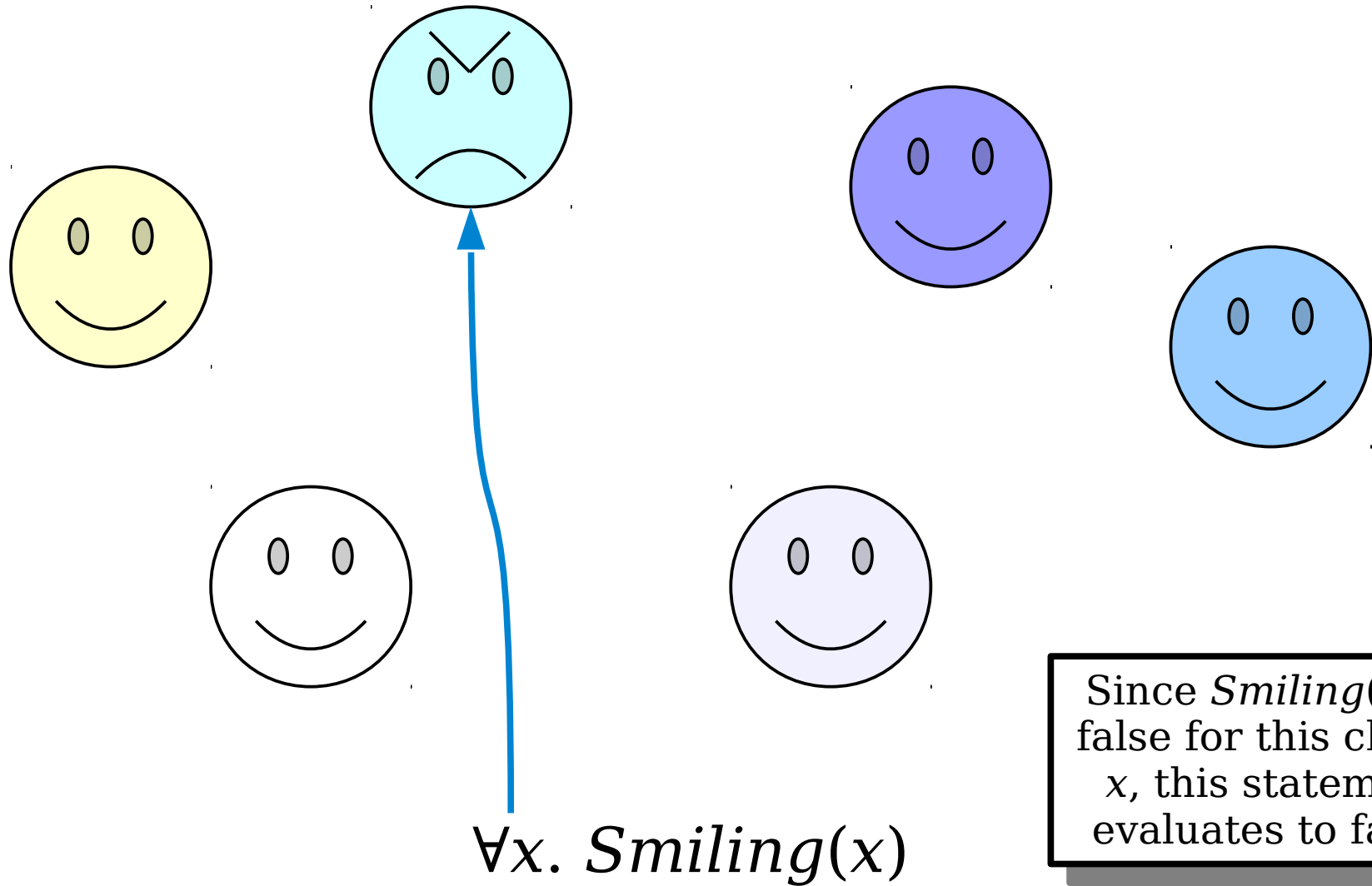
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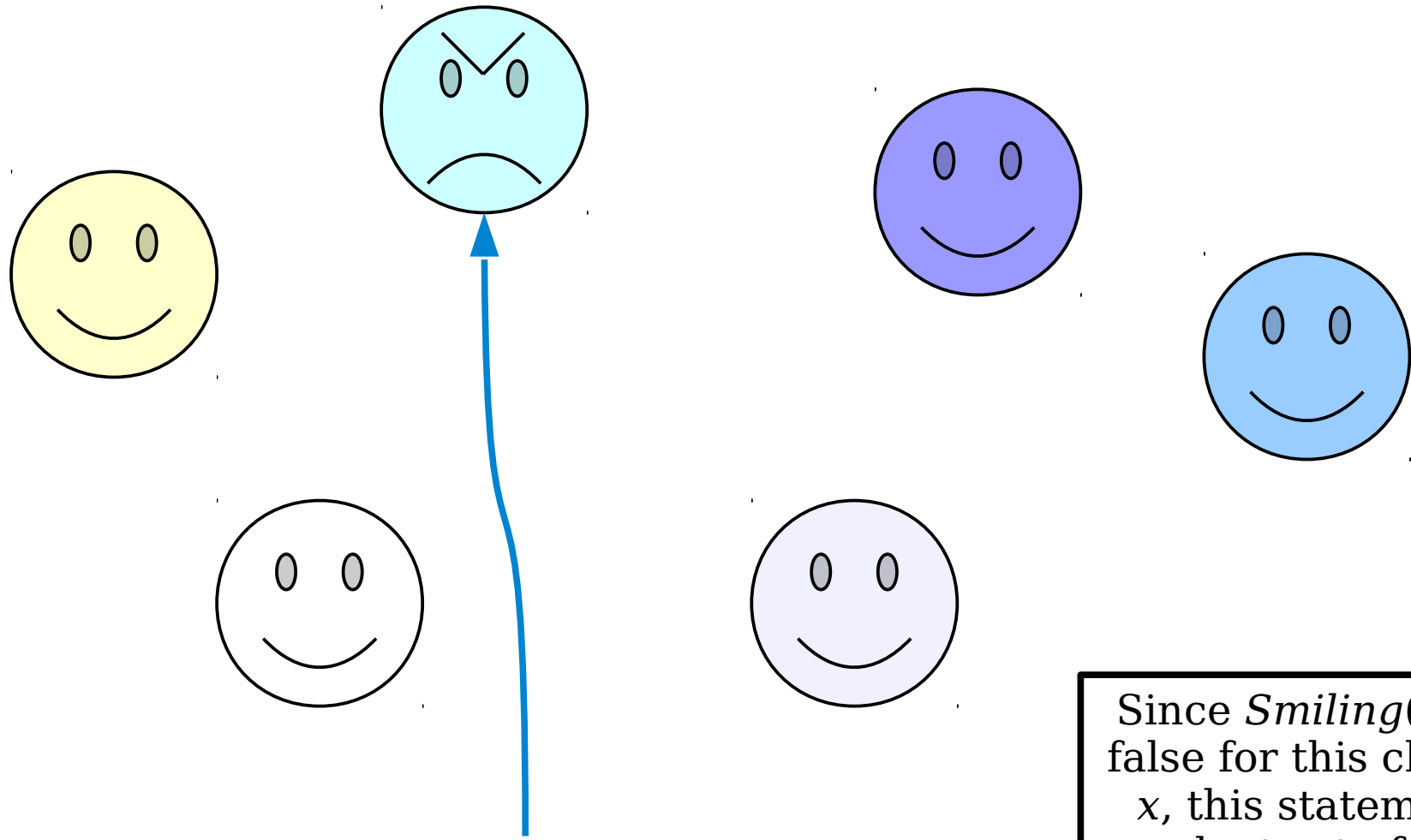
The Universal Quantifier



The Universal Quantifier



The Universal Quantifier

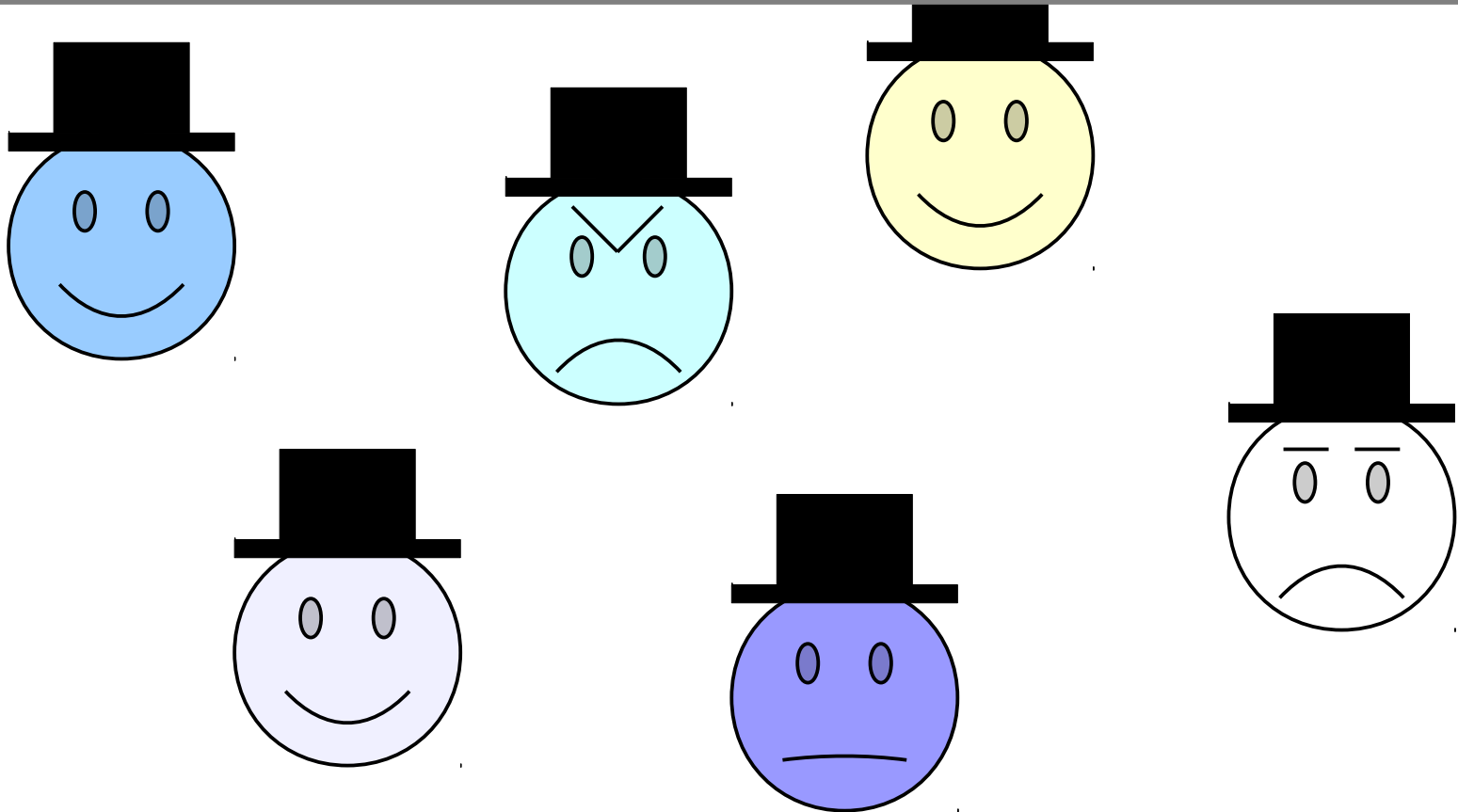


~~$\forall x. \text{Smiling}(x)$~~

Since $\text{Smiling}(x)$ is false for this choice x , this statement evaluates to false.

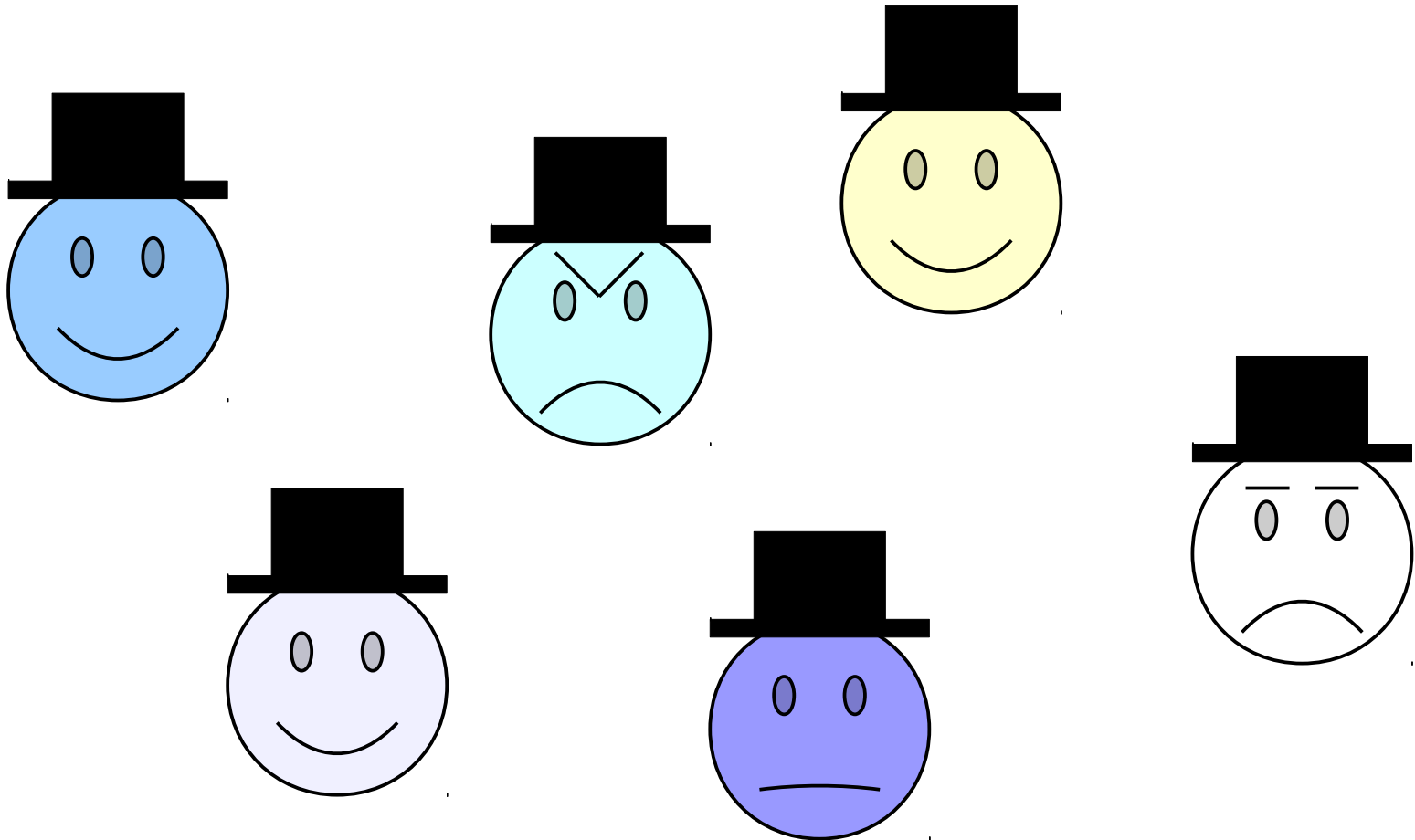
Question: In this world, is the first-order logic statement below true or false?

Respond at pollev.com/cs103



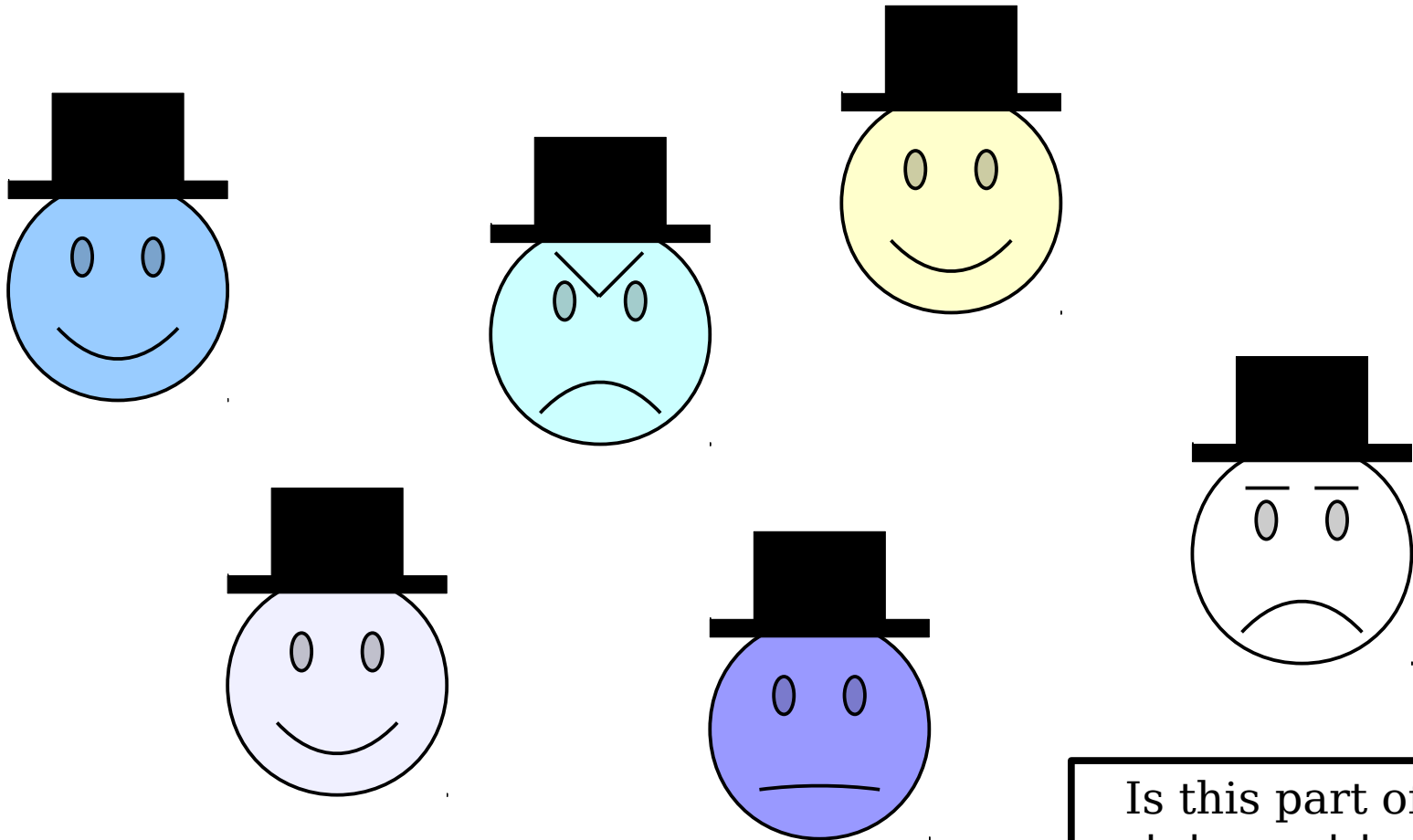
$(\forall x. Smiling(x)) \rightarrow (\forall y. WearingHat(y))$

The Universal Quantifier



$(\forall x. \textit{Smiling}(x)) \rightarrow (\forall y. \textit{WearingHat}(y))$

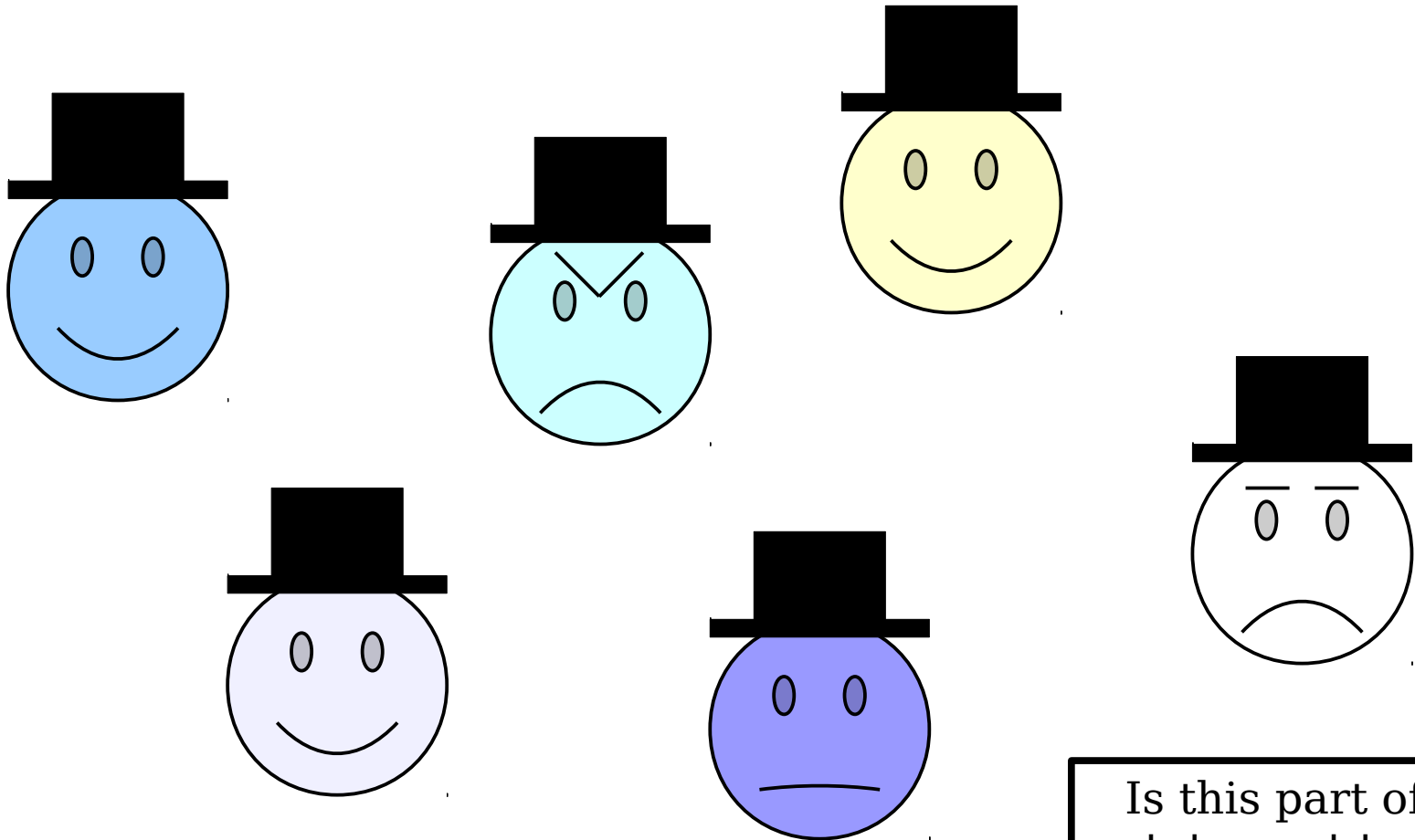
The Universal Quantifier



Is this part of the statement true or false?

$(\forall x. \textit{Smiling}(x)) \rightarrow (\forall y. \textit{WearingHat}(y))$

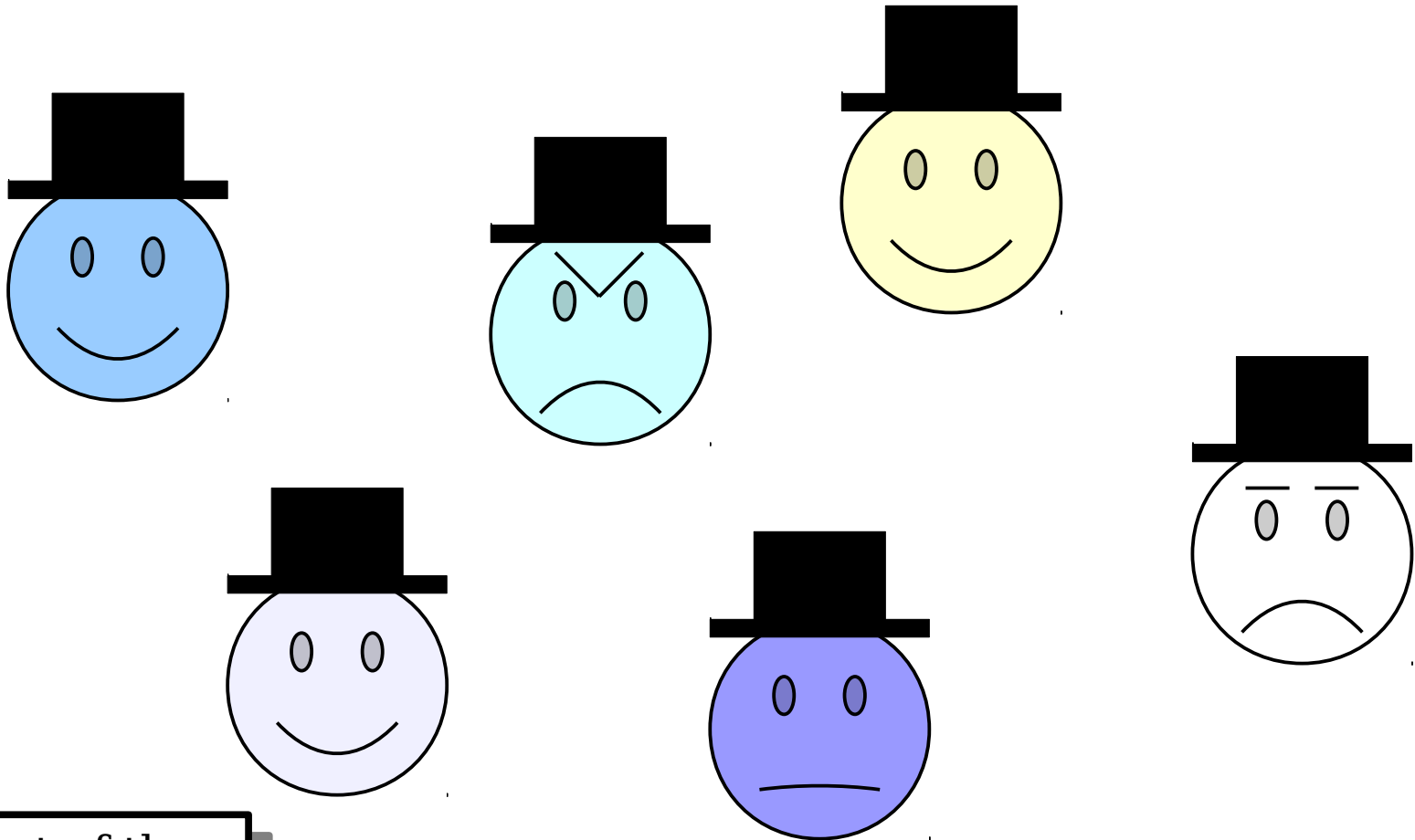
The Universal Quantifier



Is this part of the statement true or false?

$(\forall x. \textit{Smiling}(x)) \rightarrow (\forall y. \textit{WearingHat}(y))$

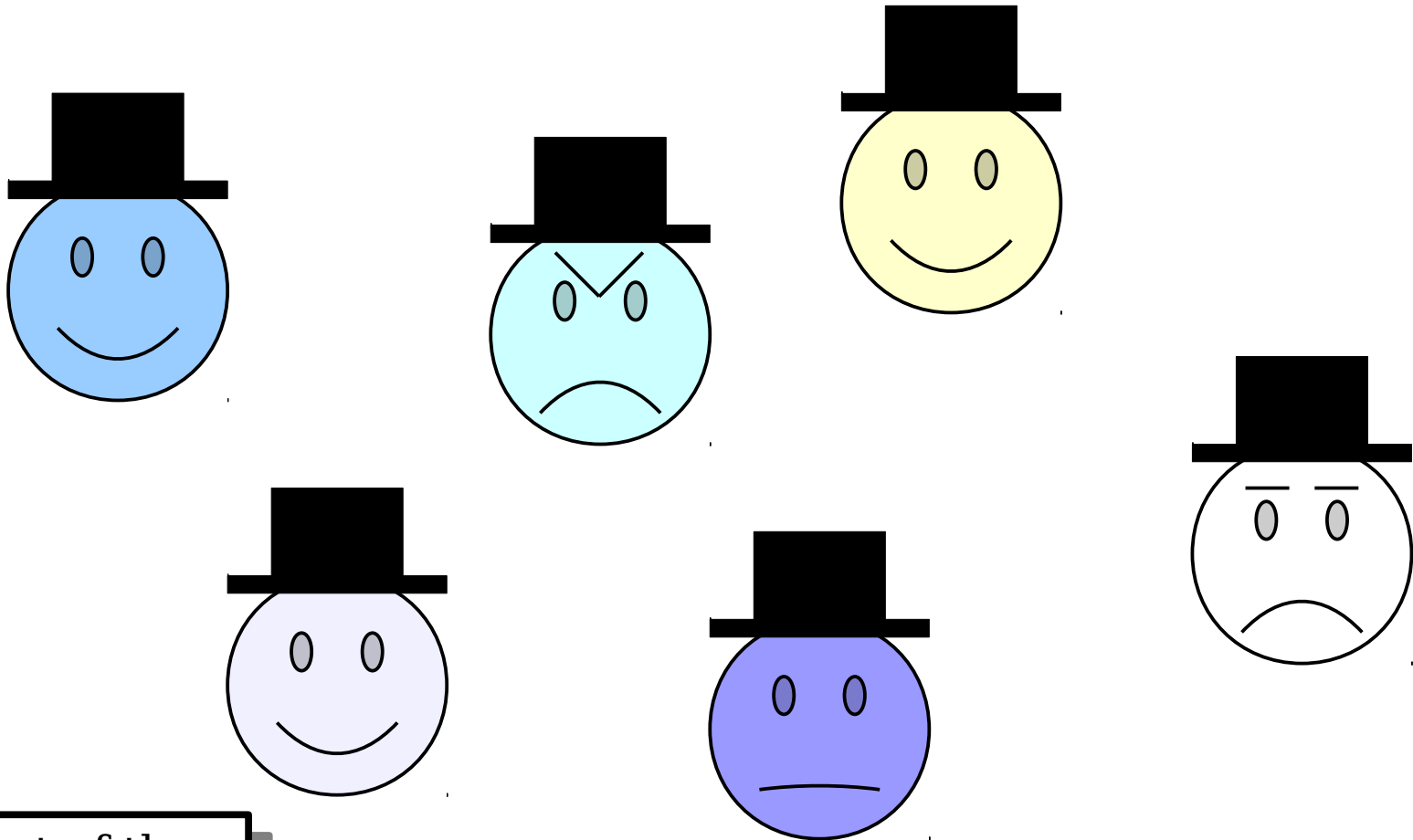
The Universal Quantifier



Is this part of the statement true or false?

$(\forall x. \textit{Smiling}(x)) \rightarrow (\forall y. \textit{WearingHat}(y))$

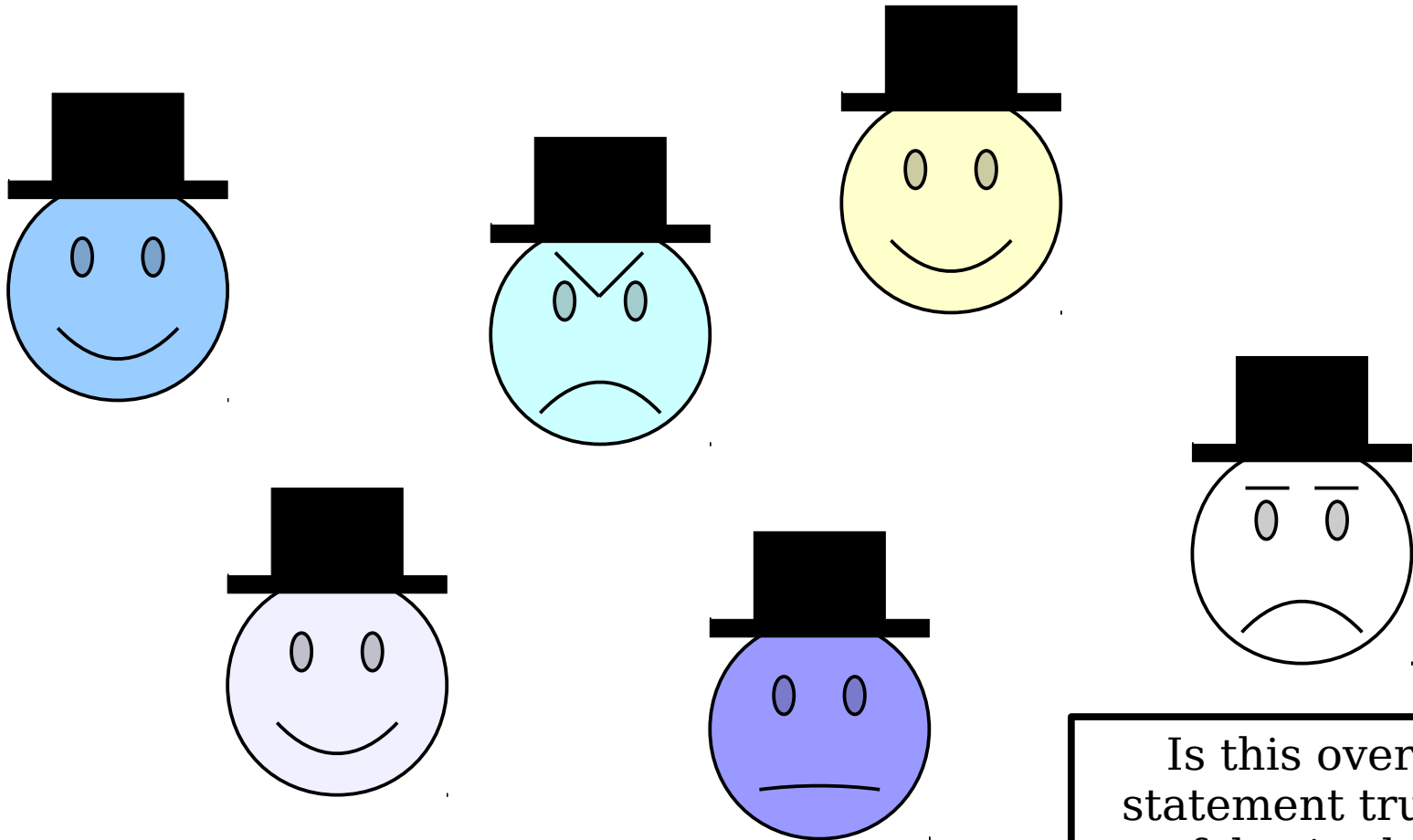
The Universal Quantifier



Is this part of the statement true or false?

~~$(\forall x. \textit{Smiling}(x))$~~ \rightarrow $(\forall y. \textit{WearingHat}(y))$

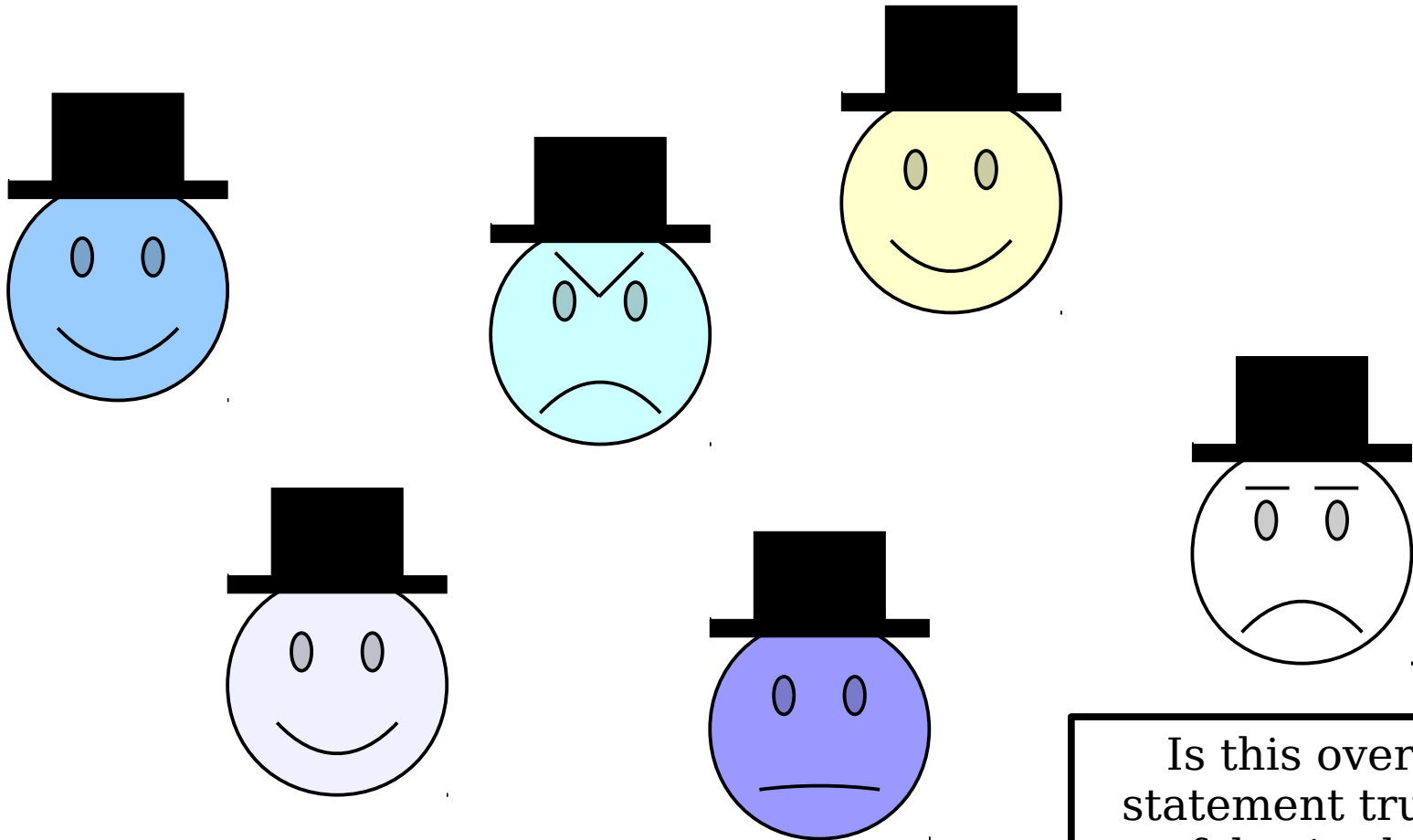
The Universal Quantifier



Is this overall statement true or false in this scenario?

~~$(\forall x. \textit{Smiling}(x))$~~ $\rightarrow (\forall y. \textit{WearingHat}(y))$

The Universal Quantifier



Is this overall statement true or false in this scenario?

$(\forall x. \textit{Smiling}(x)) \rightarrow (\forall y. \textit{WearingHat}(y))$

Fun with Edge Cases

$\forall x. \textit{Smiling}(x)$

Fun with Edge Cases

Universally-quantified statements are said to be *vacuously true* in empty worlds.

$\forall x. \textit{Smiling}(x)$

Let's take a quick break!

Translating into First-Order Logic

Translating Into Logic

- First-order logic is an excellent tool for manipulating definitions and theorems to learn more about them.
- Need to take a negation? Translate your statement into FOL, negate it, then translate it back.
- Want to prove something by contrapositive? Translate your implication into FOL, take the contrapositive, then translate it back.

Translating Into Logic

- When translating from English into first-order logic, we recommend that you ***think of first-order logic as a mathematical programming language.***
- Your goal is to learn how to combine basic concepts (quantifiers, connectives, etc.) together in ways that say what you mean.

Using the predicates

- *Smiling*(x), which states that x is smiling, and
- *WearingHat*(x), which states that x is wearing a hat,

write a sentence in first-order logic that says

some smiling person wears a hat.

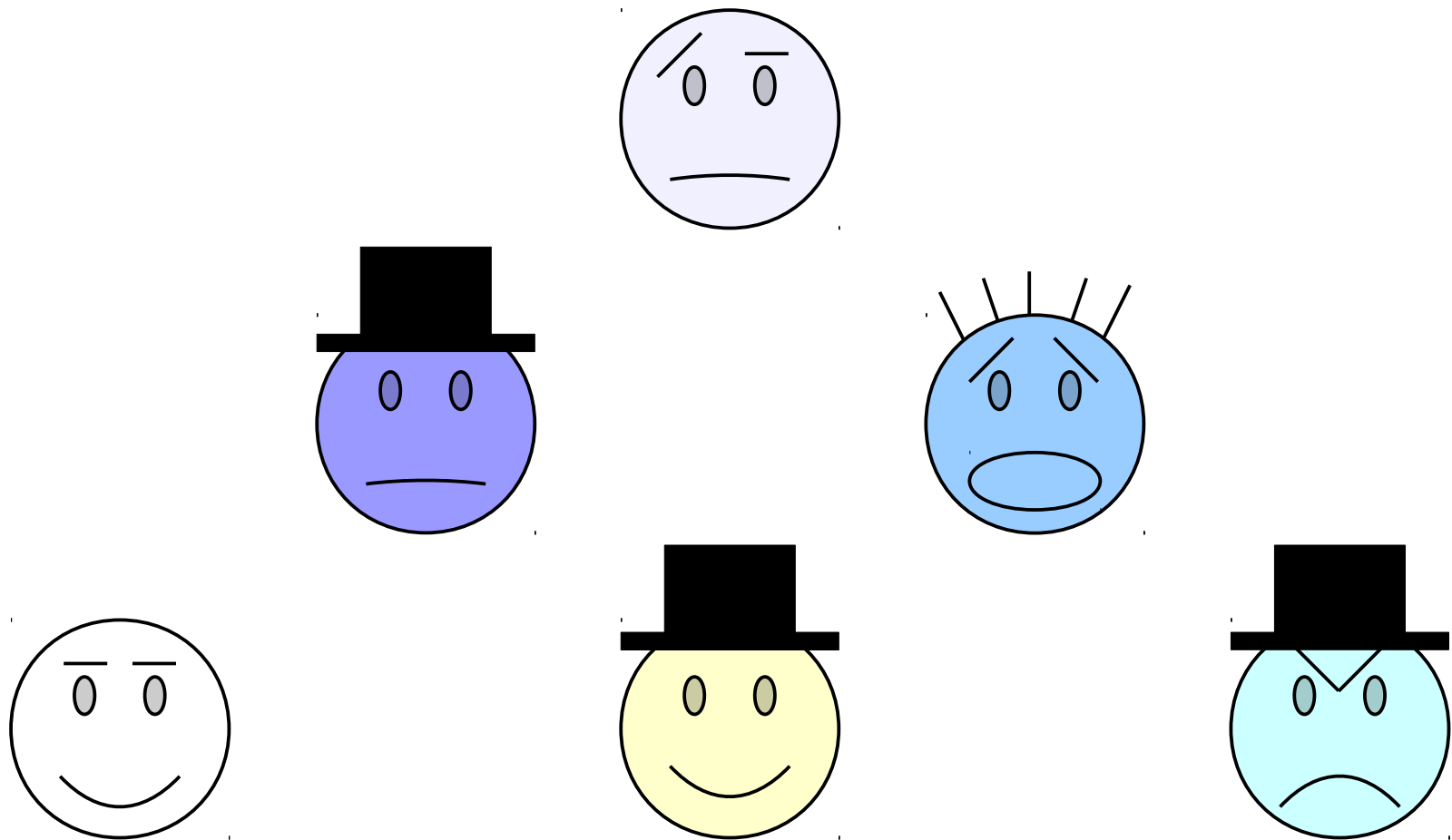
Try it yourself: Give this your best shot – it's okay if you're not sure!

Respond at pollev.com/cs103

“Some smiling person wears a hat.”

$\exists x. (\textit{Smiling}(x) \wedge \textit{WearingHat}(x))$

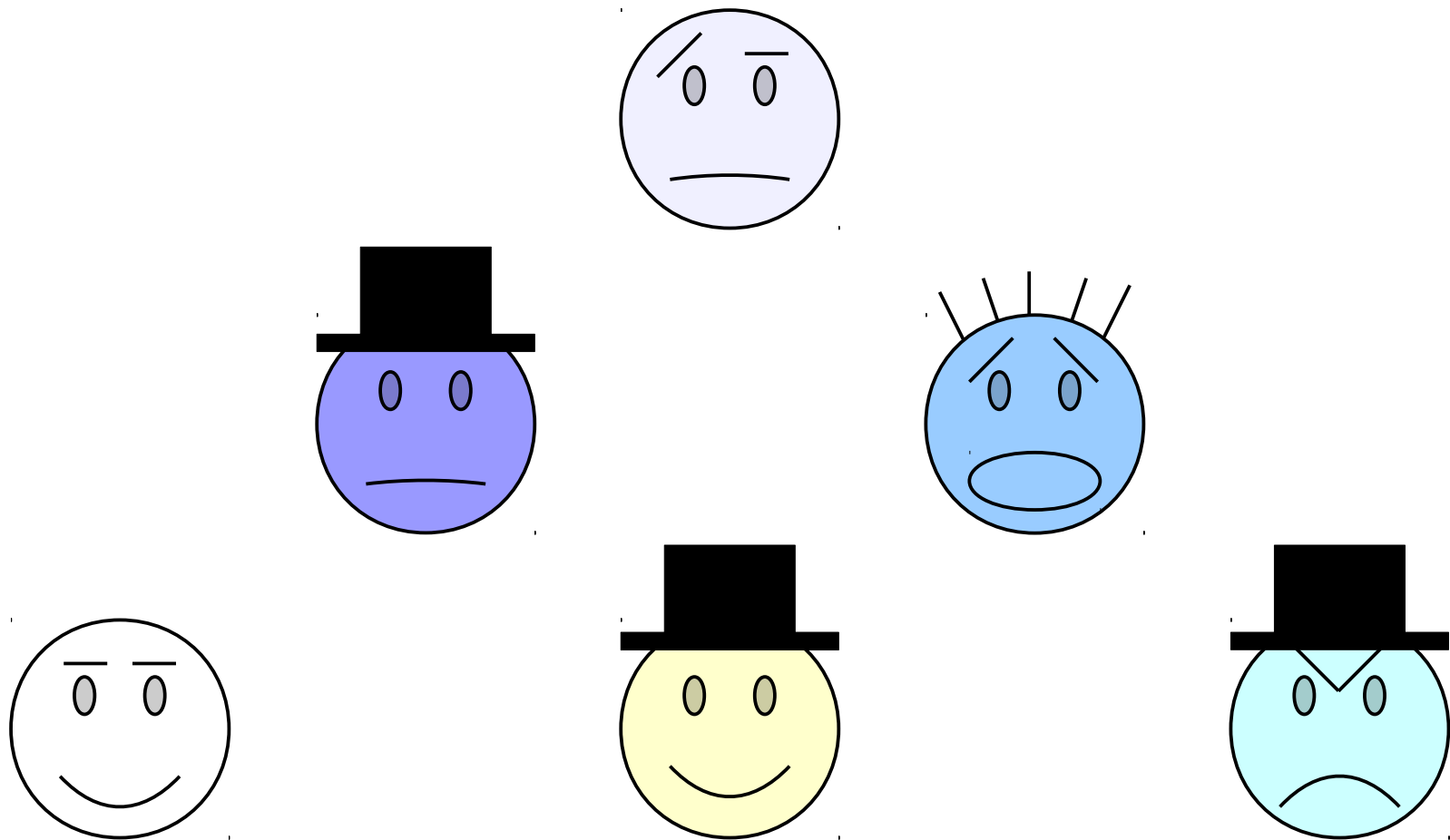
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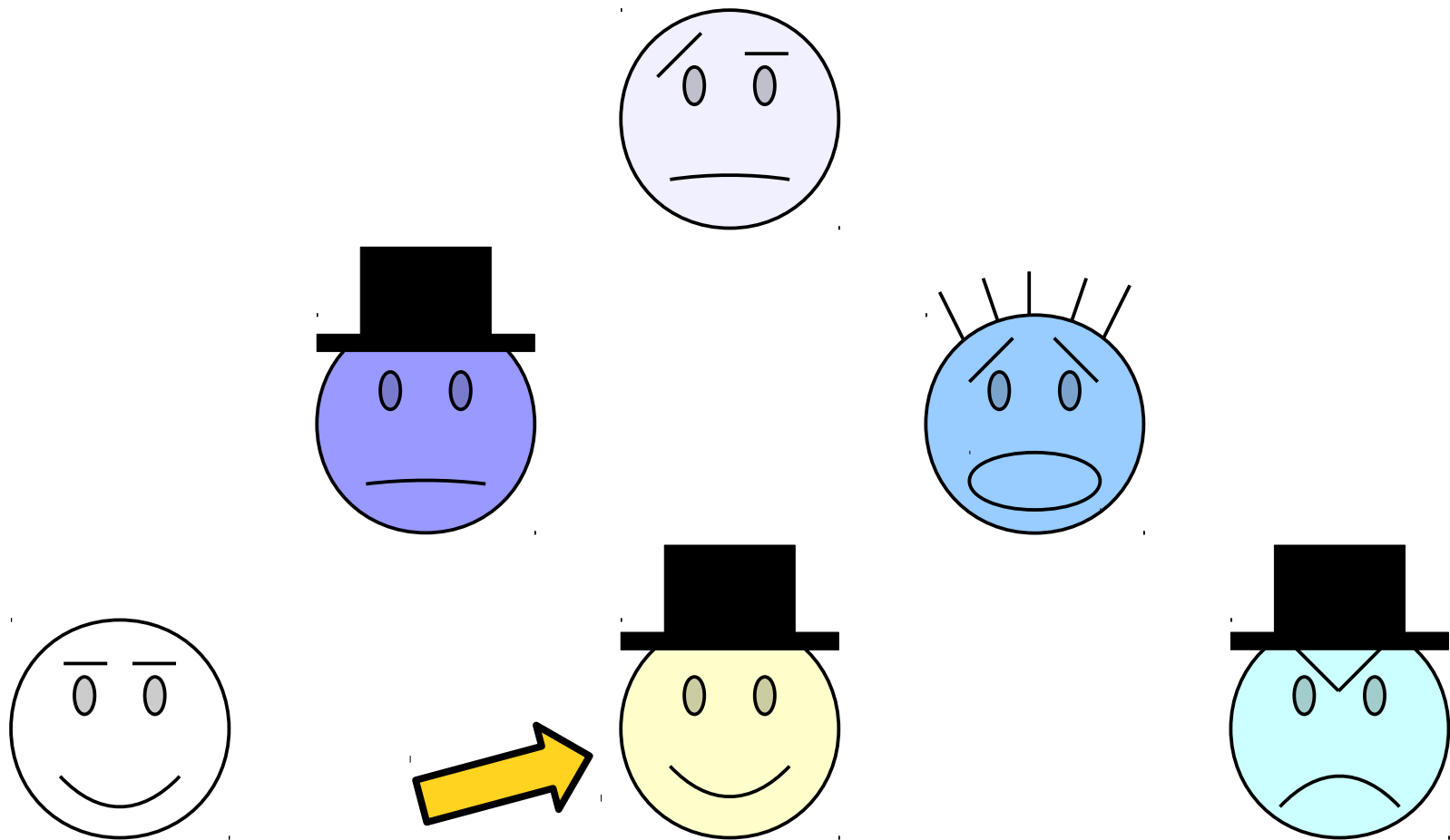
$\exists x. (Smiling(x) \rightarrow WearingHat(x))$



“Some smiling person wears a hat.”

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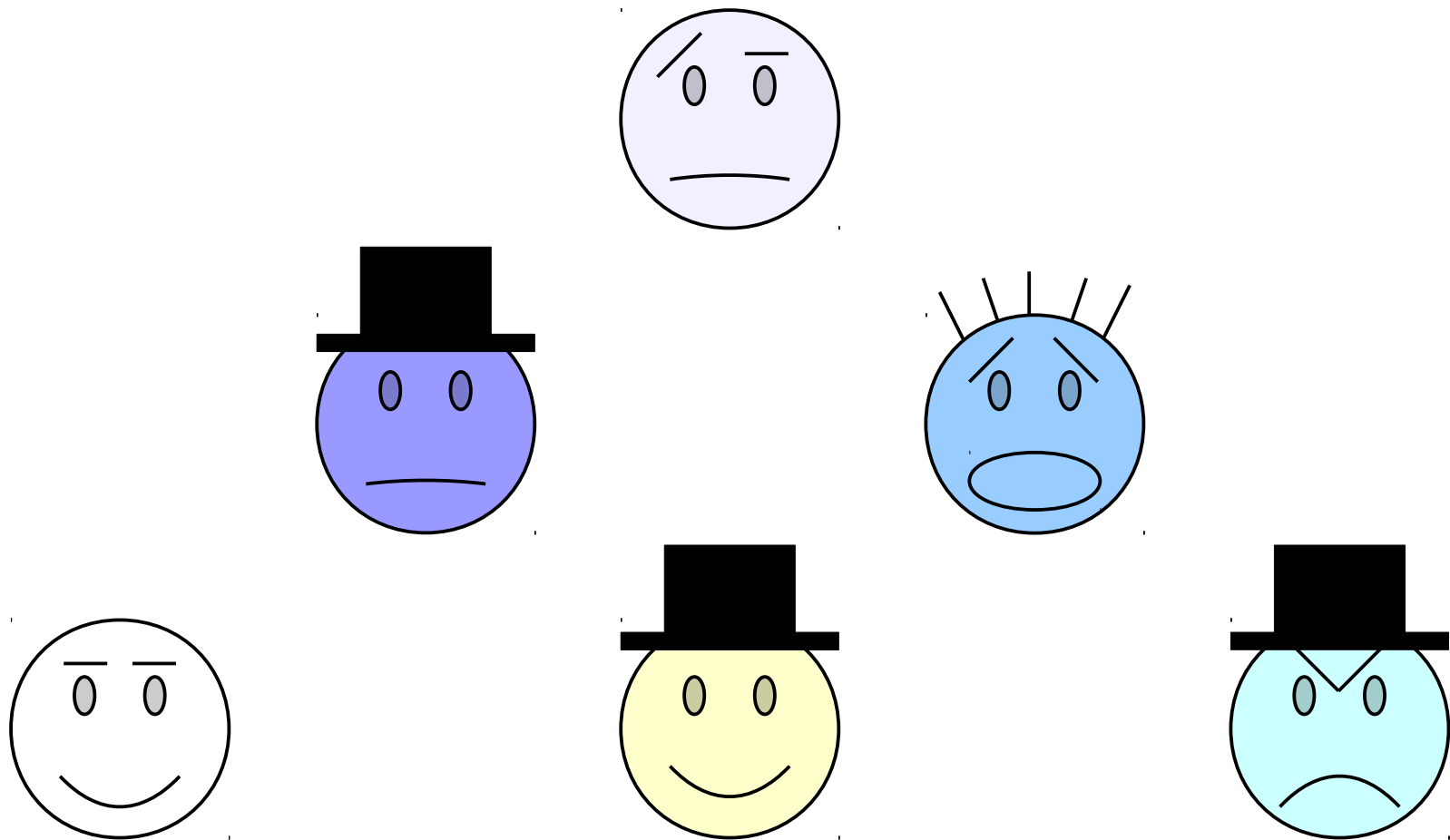
$\exists x. (Smiling(x) \rightarrow WearingHat(x))$



“Some smiling person wears a hat.” *True*

$\exists x. (Smiling(x) \wedge WearingHat(x))$

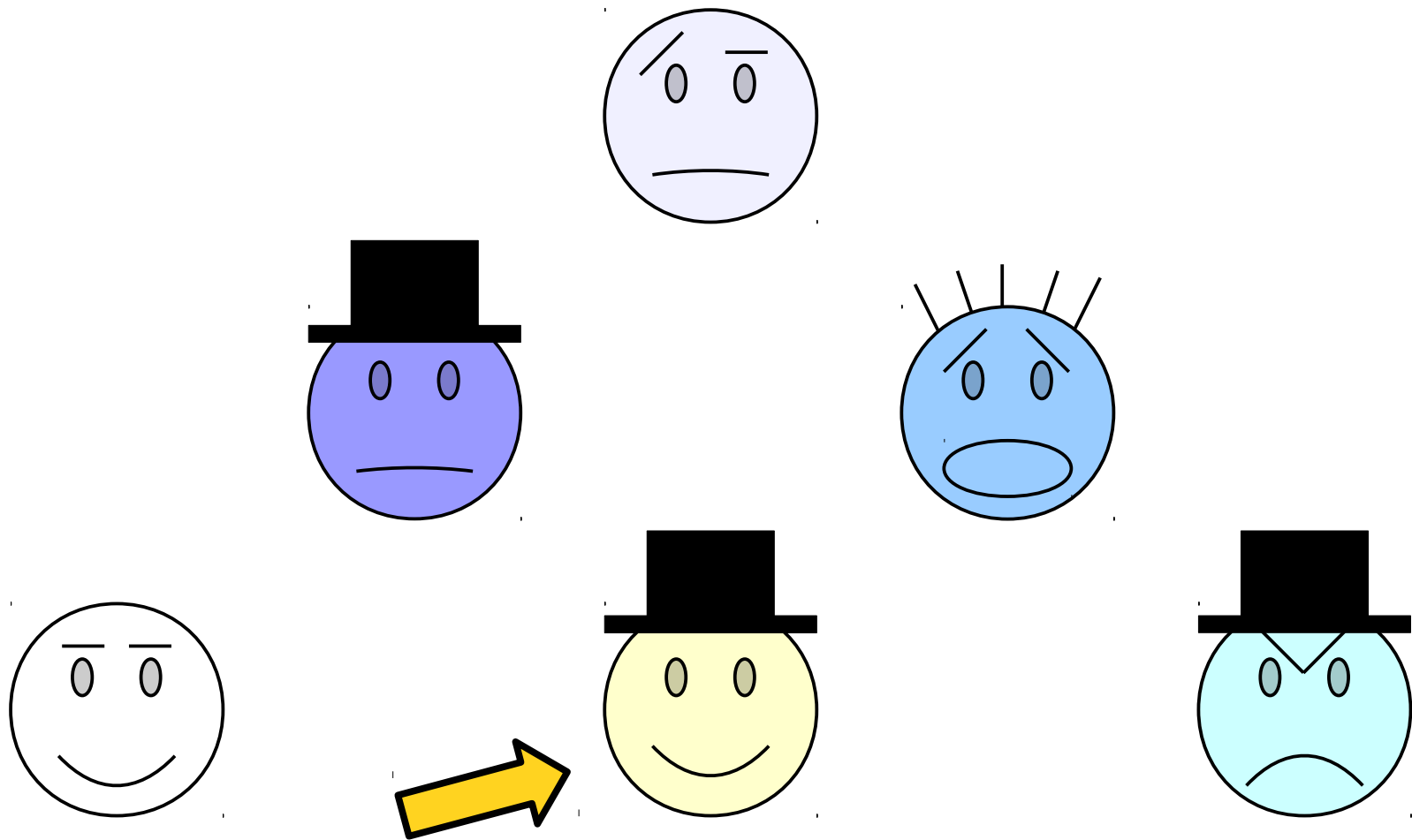
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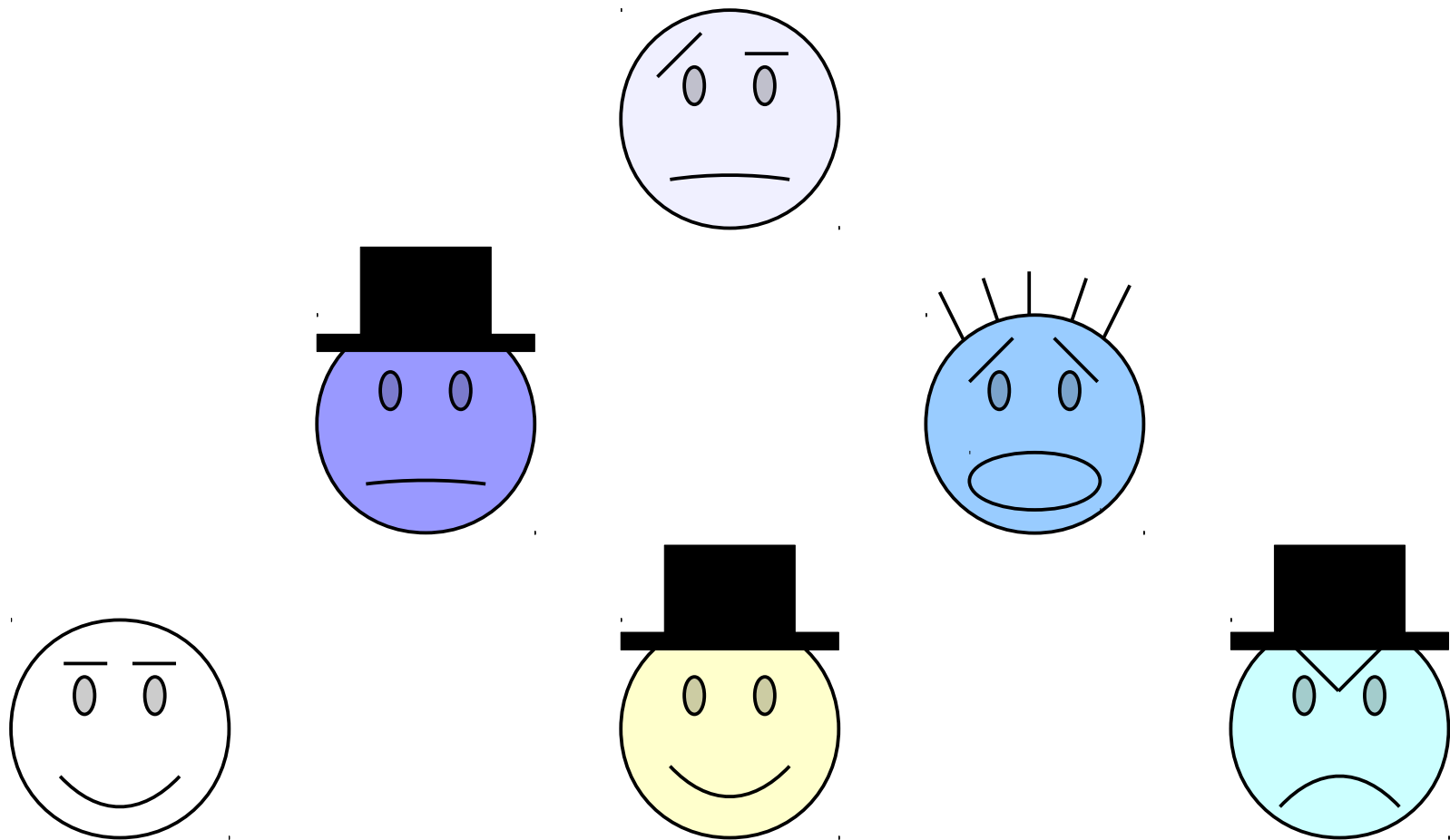
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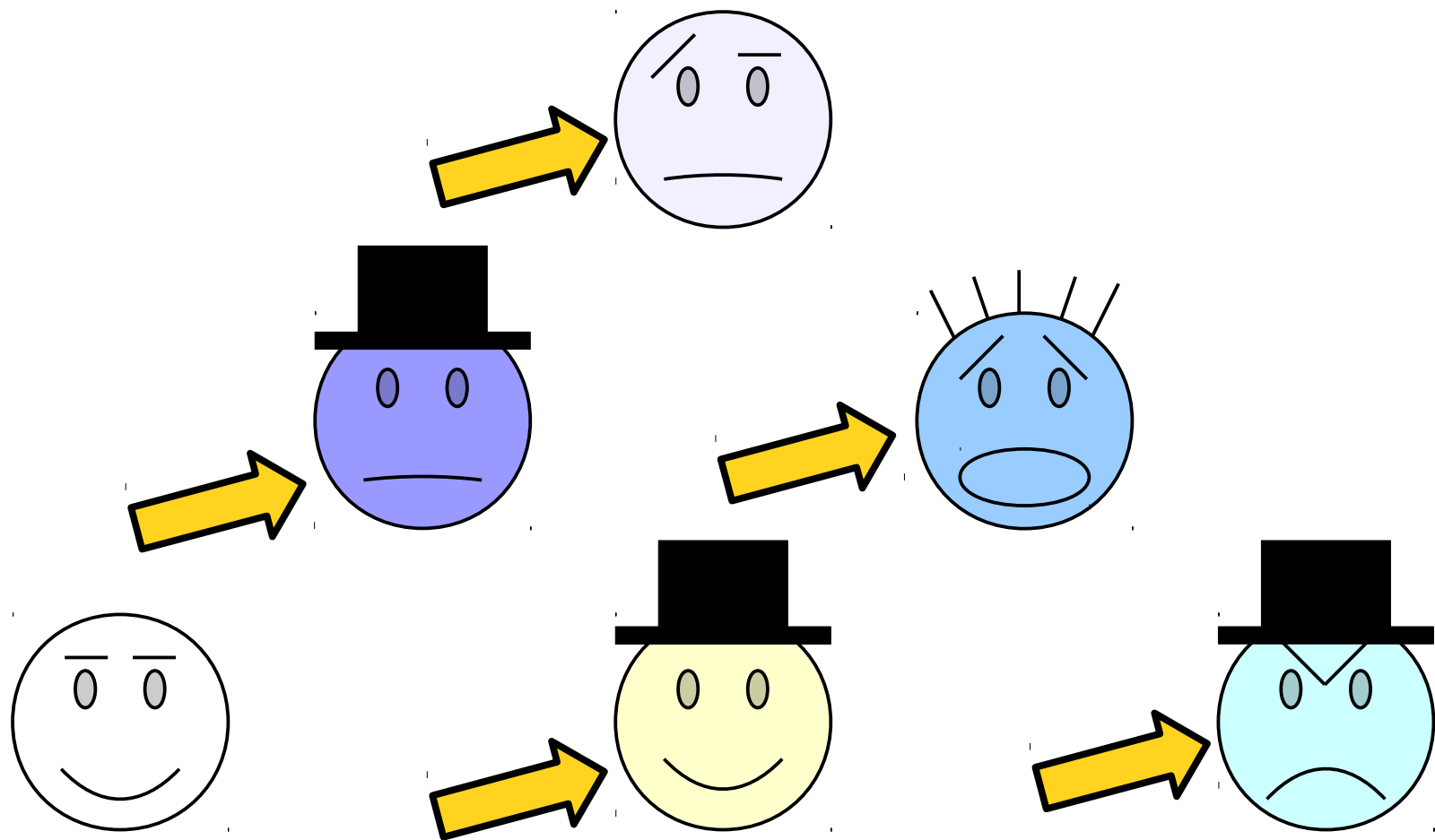
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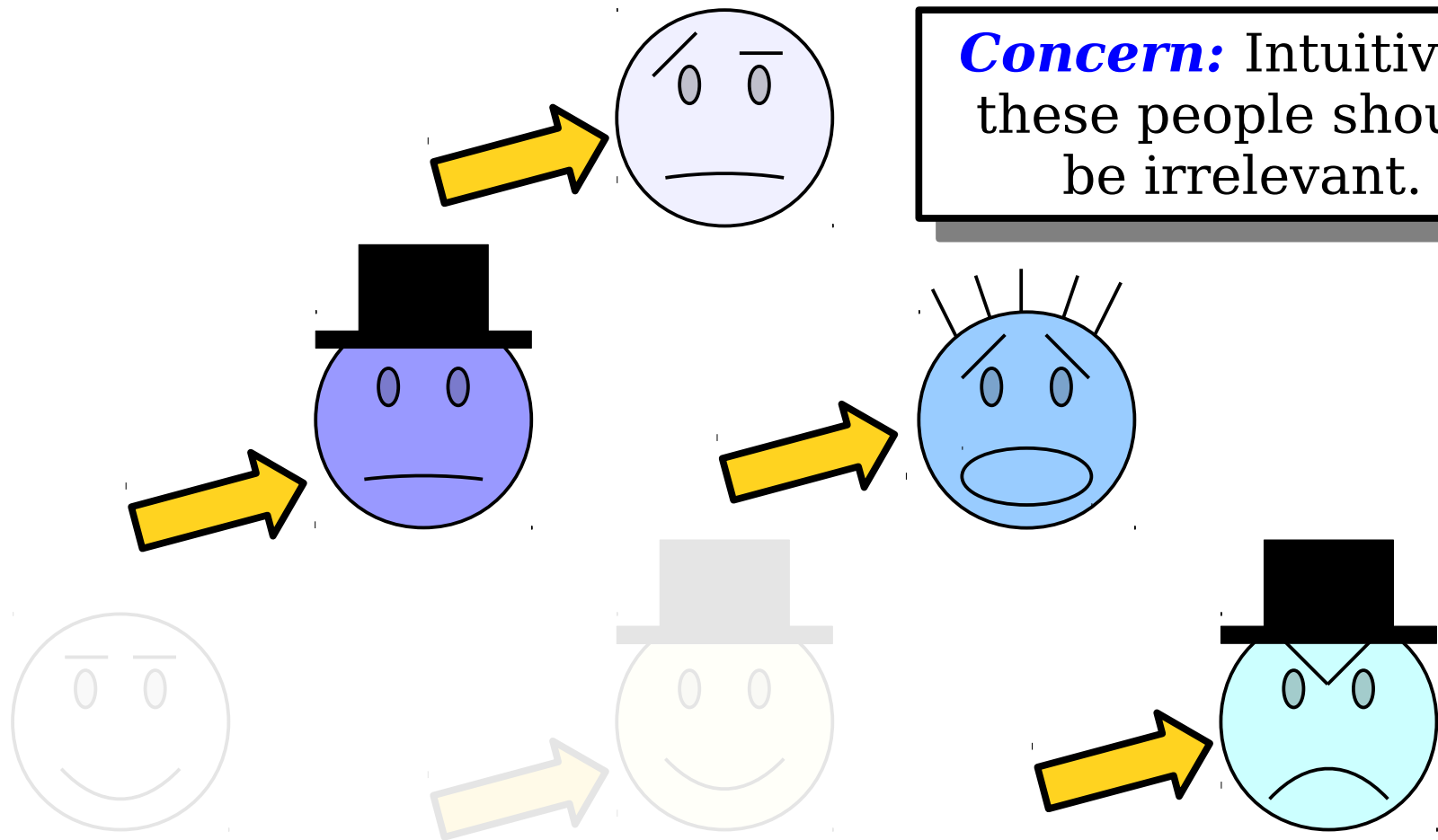
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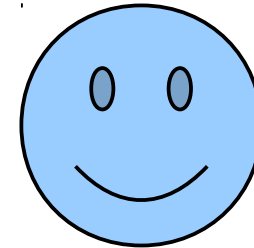
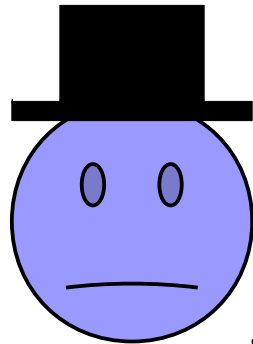
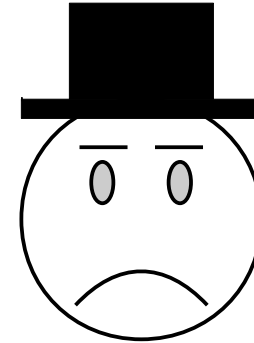
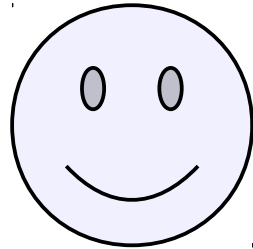
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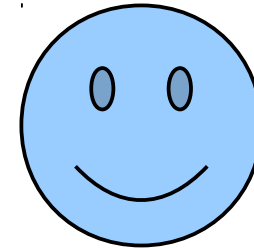
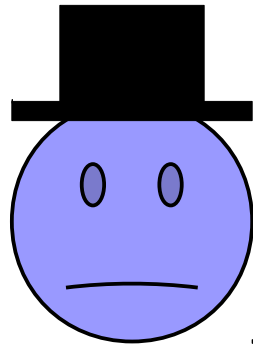
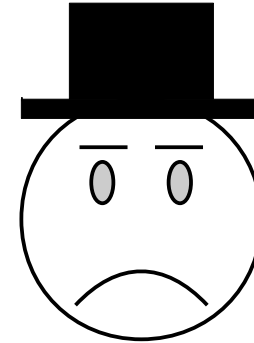
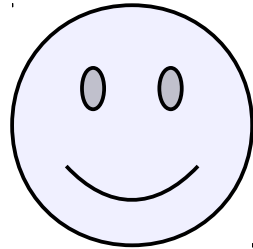
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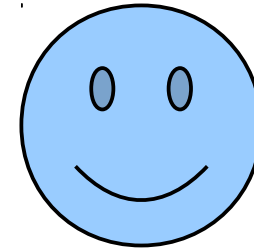
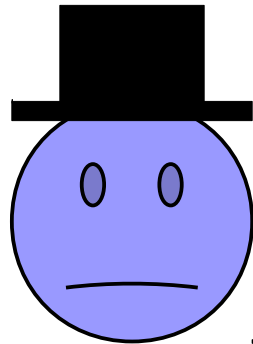
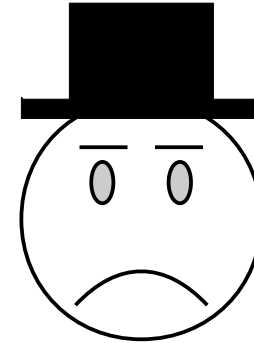
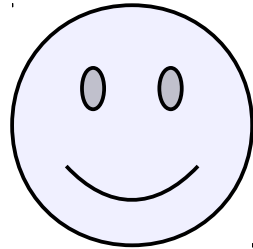
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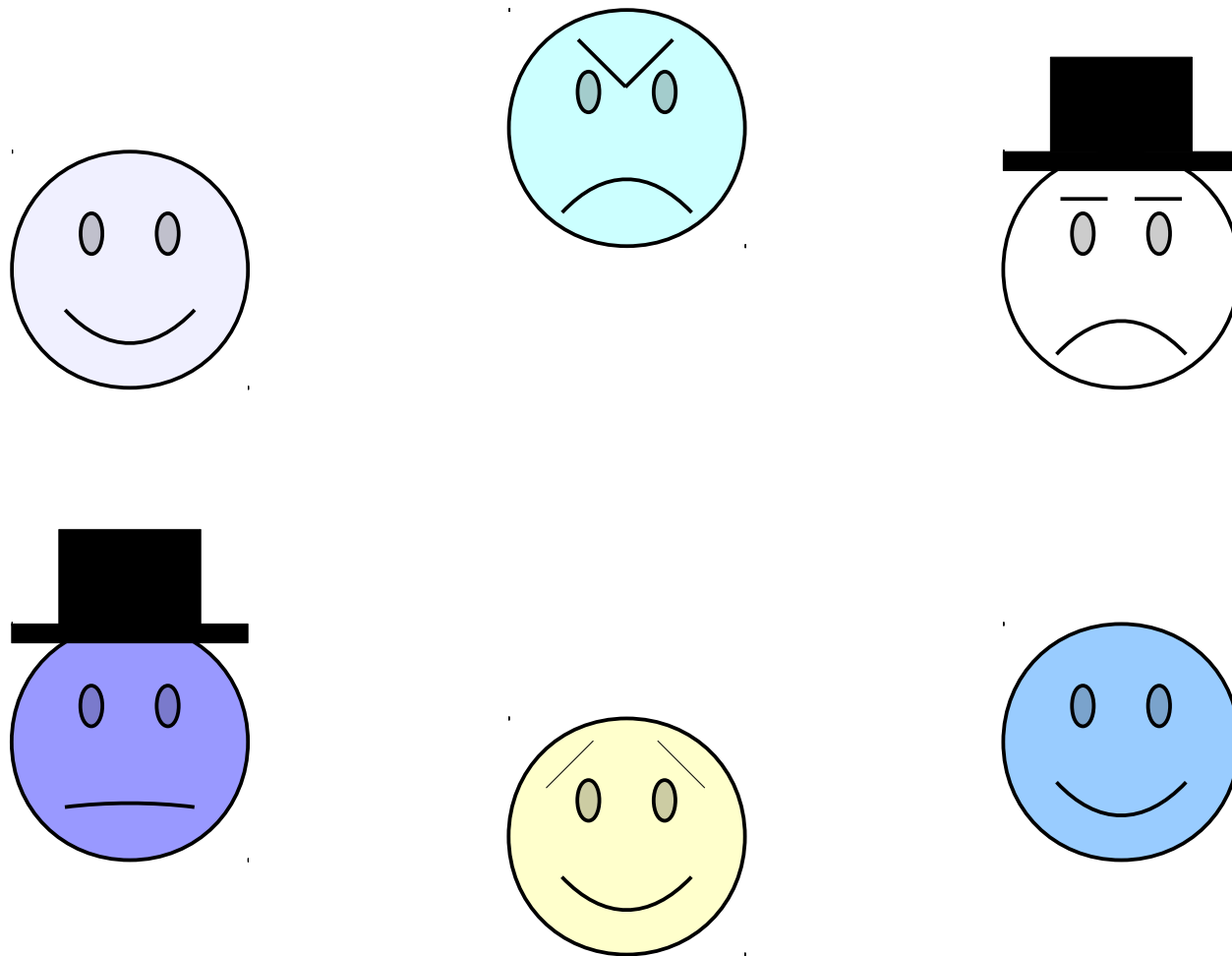
$\exists x. (Smiling(x) \rightarrow WearingHat(x))$



“Some smiling person wears a hat.” *False*

$\exists x. (Smiling(x) \wedge WearingHat(x))$

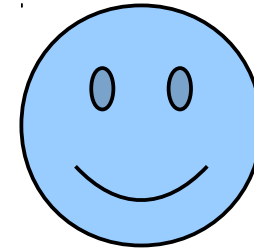
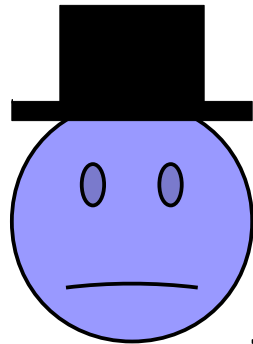
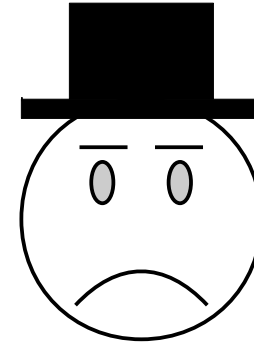
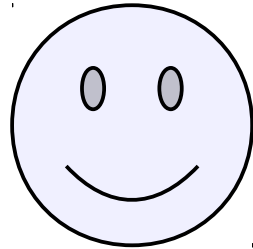
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“Some smiling person wears a hat.” *False*

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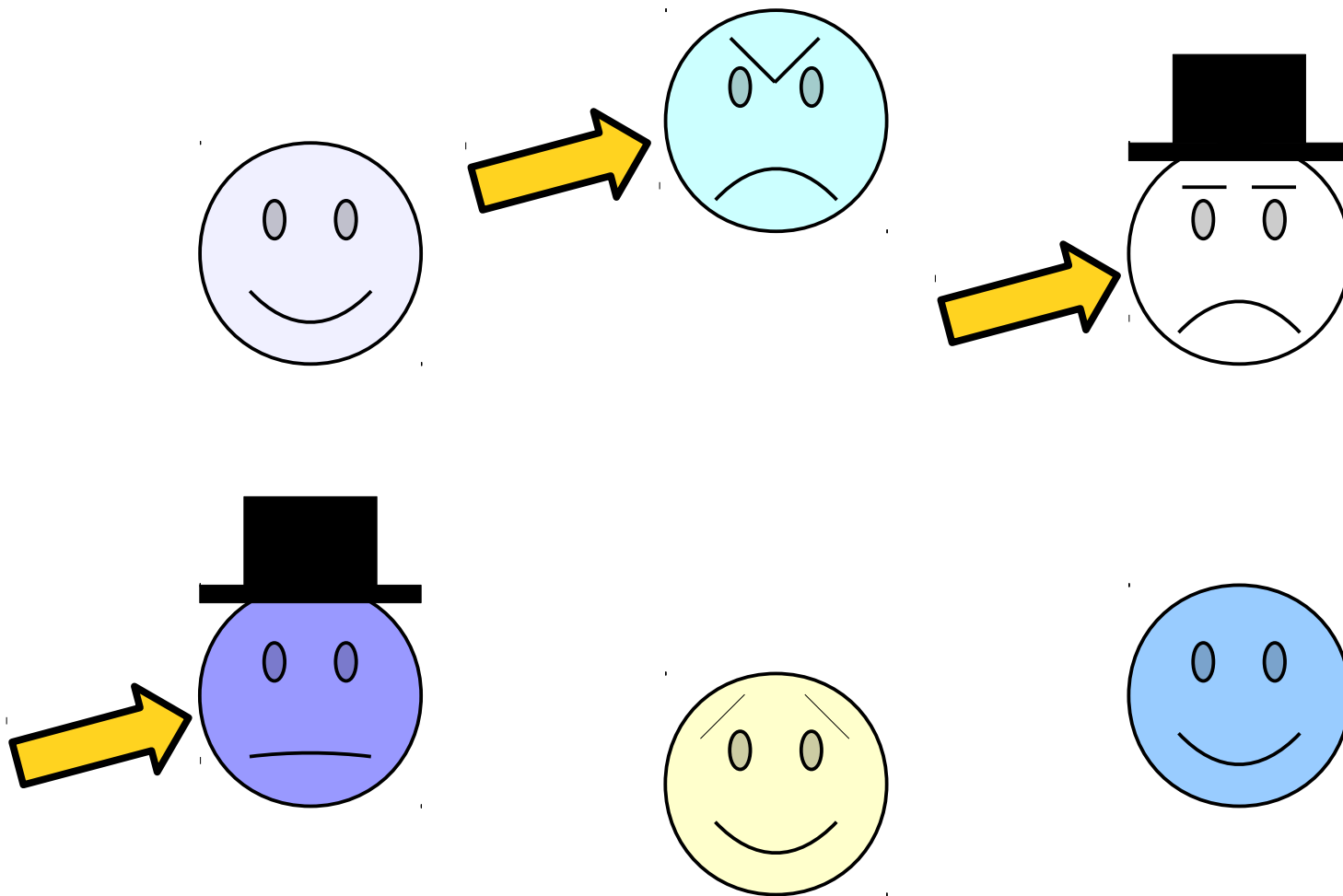
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“Some smiling person wears a hat.” ***False***

$\exists x. (Smiling(x) \wedge WearingHat(x))$ ***False***

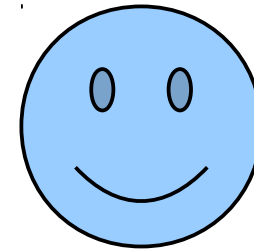
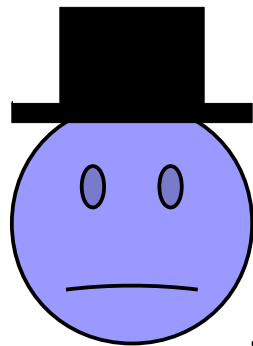
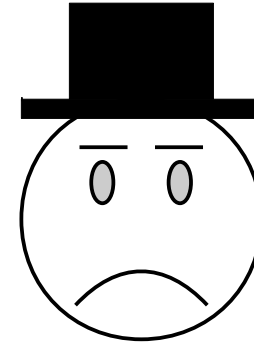
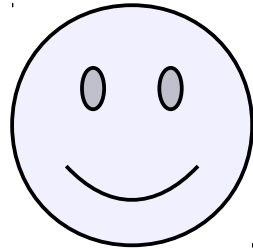
$\exists x. (Smiling(x) \rightarrow WearingHat(x))$



“Some smiling person wears a hat.” **False**

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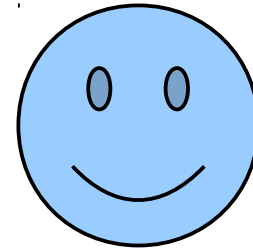
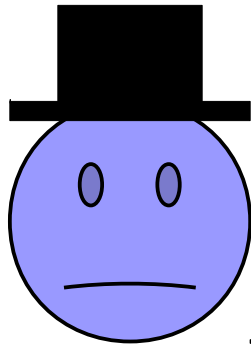
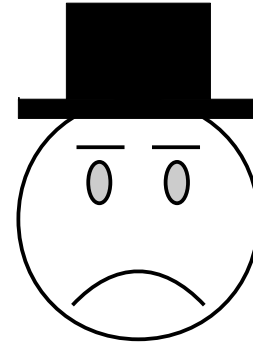
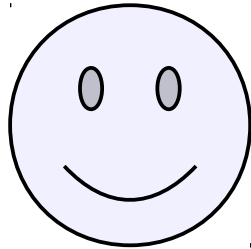
$\exists x. (Smiling(x) \rightarrow WearingHat(x))$ **True**



“Some smiling person wears a hat.” ***False***

$\exists x. (Smiling(x) \wedge WearingHat(x))$ ***False***

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“Some smiling person wears a hat.” ***False***

$\exists x. (Smiling(x) \wedge WearingHat(x))$ ***False***

~~$\exists x. (Smiling(x) \rightarrow WearingHat(x))$~~ ***True***

“Some P is a Q ”

translates as

$\exists x. (P(x) \wedge Q(x))$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x. (P(x) \wedge Q(x))$$

If x is an example, it must have property P on top of property Q .

Using the predicates

- *Smiling*(x), which states that x is smiling, and
- *WearingHat*(x), which states that x is wearing a hat,

write a sentence in first-order logic that says

every smiling person wears a hat.

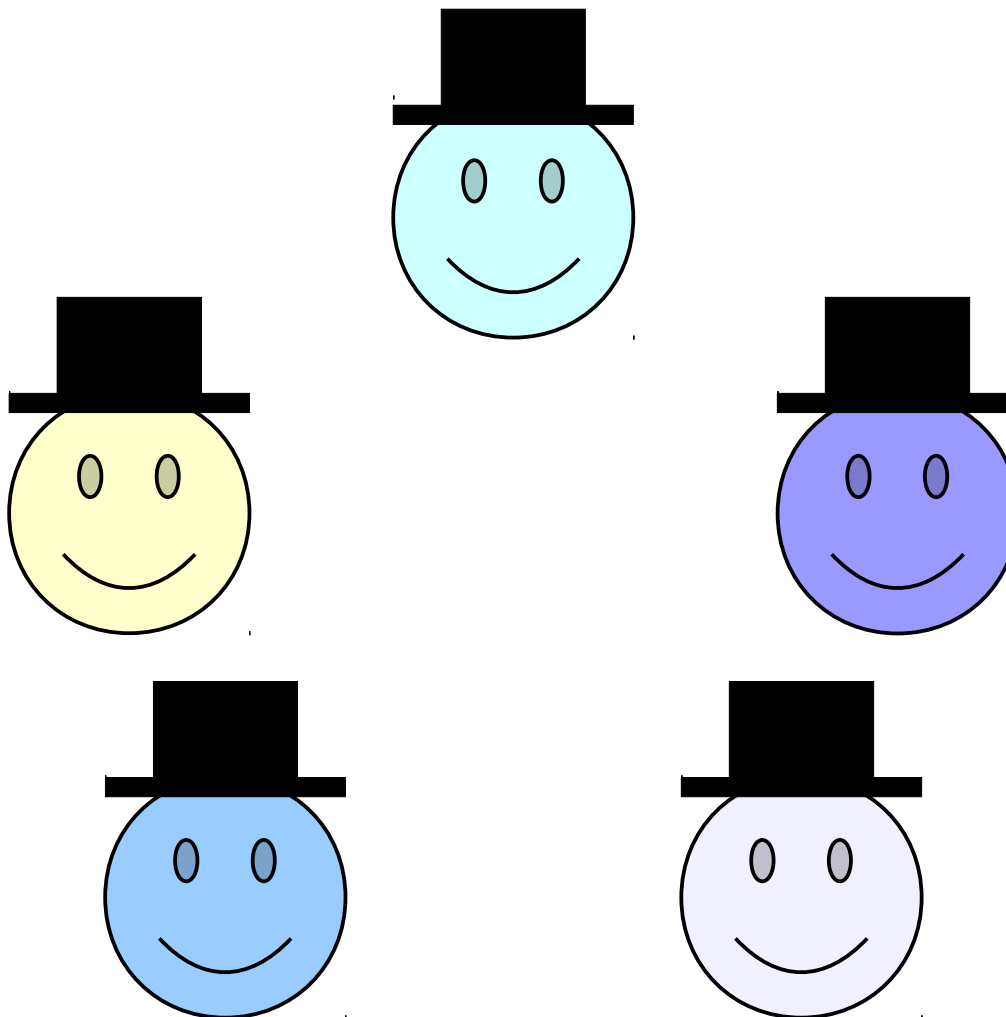
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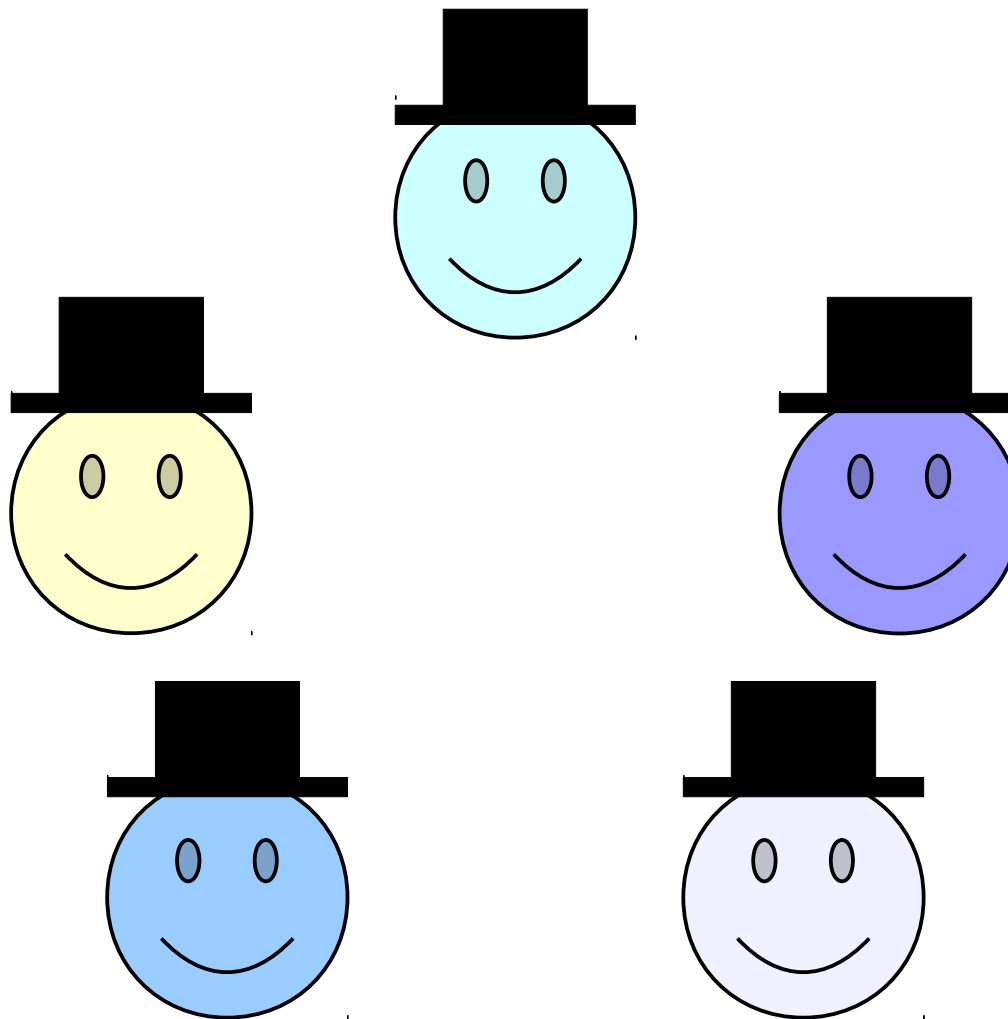
$\forall x. (\textit{Smiling}(x) \rightarrow \textit{WearingHat}(x))$



“Every smiling person wears a hat.”

$\forall x. (Smiling(x) \wedge WearingHat(x))$

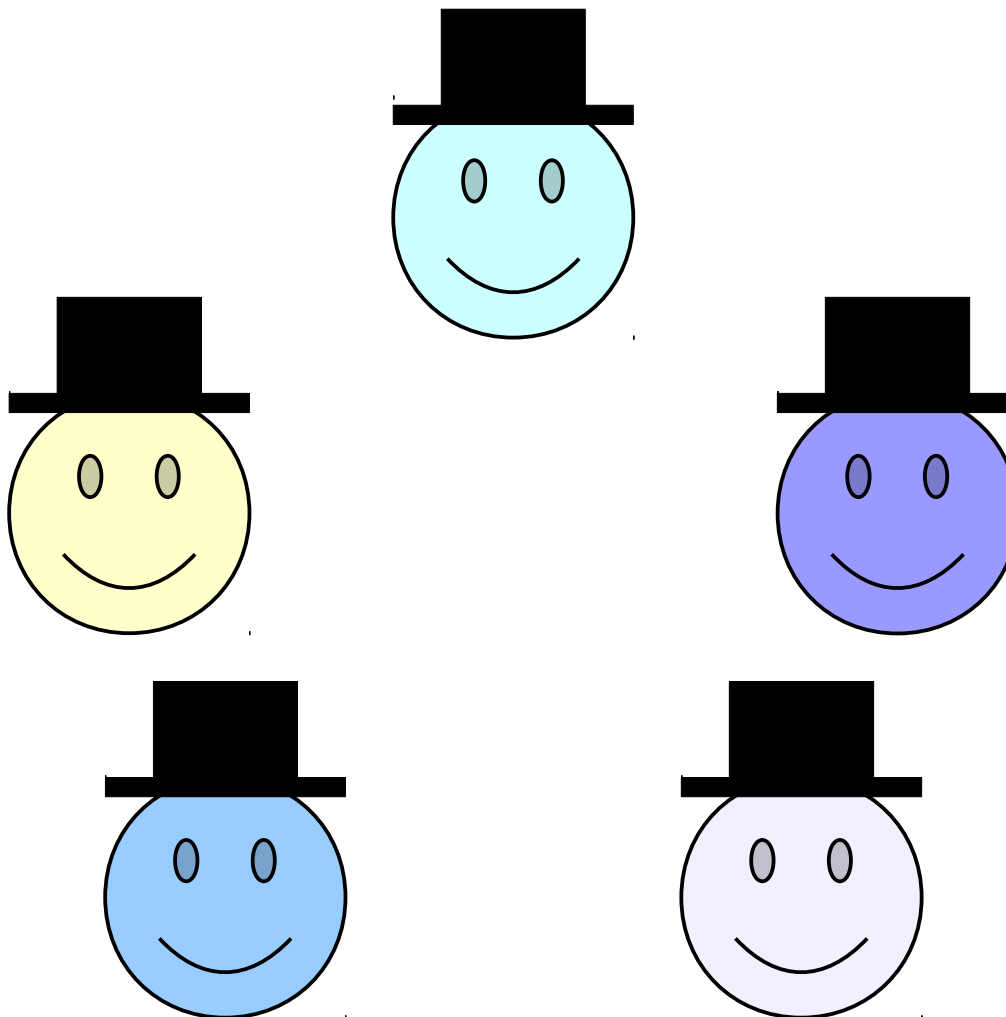
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“Every smiling person wears a hat.” *True*

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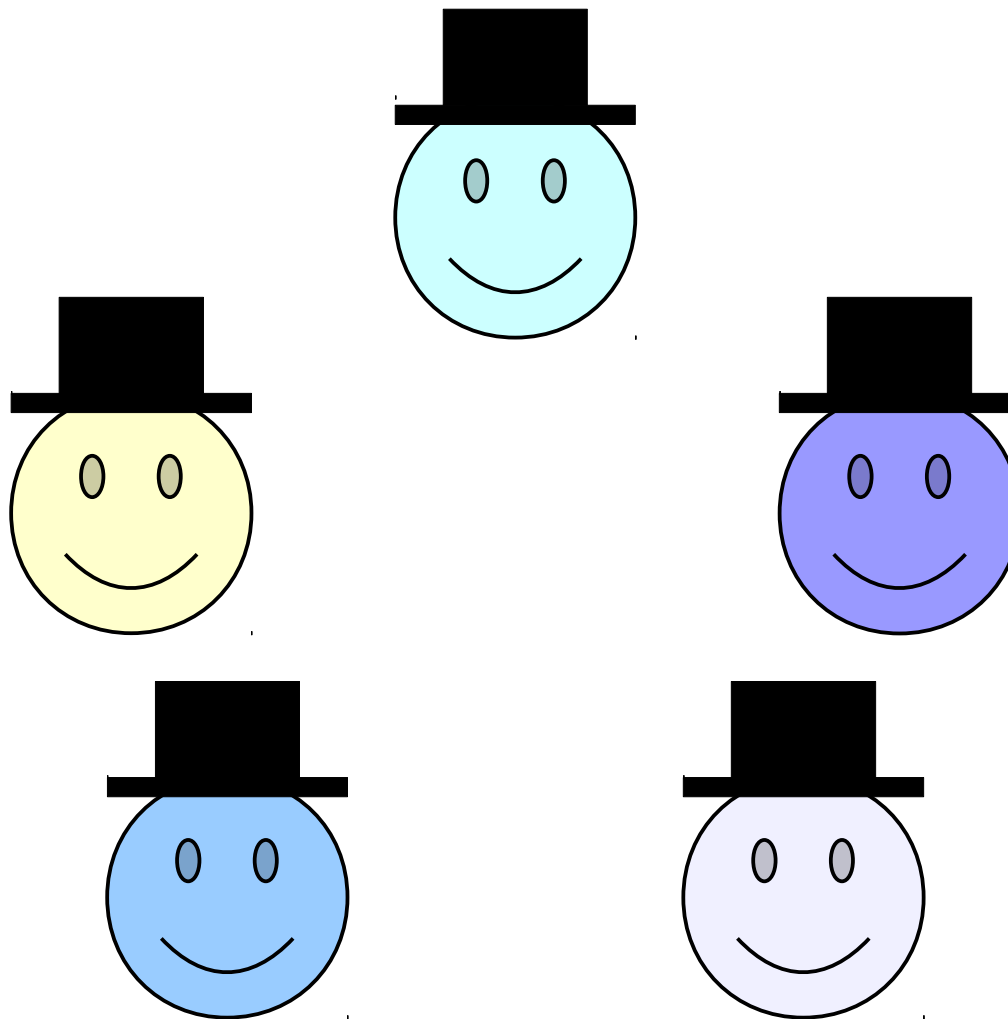
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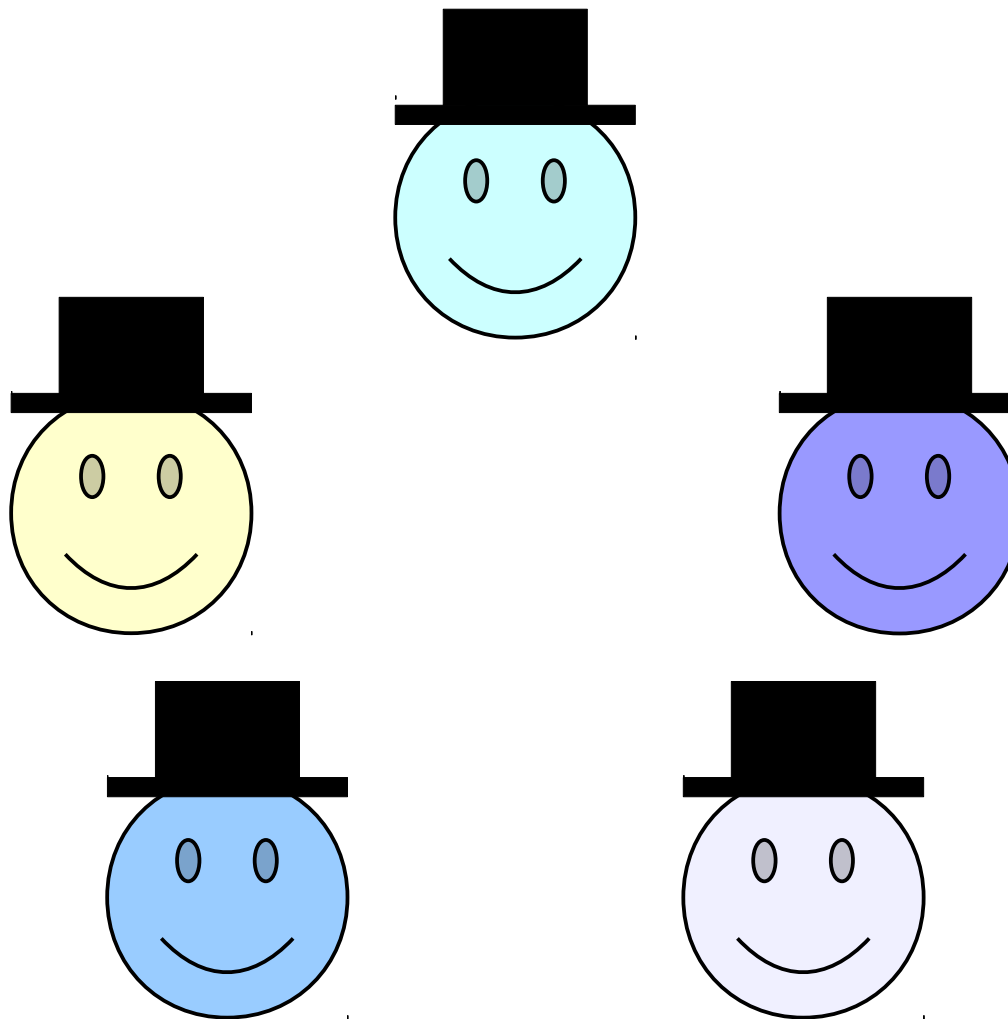
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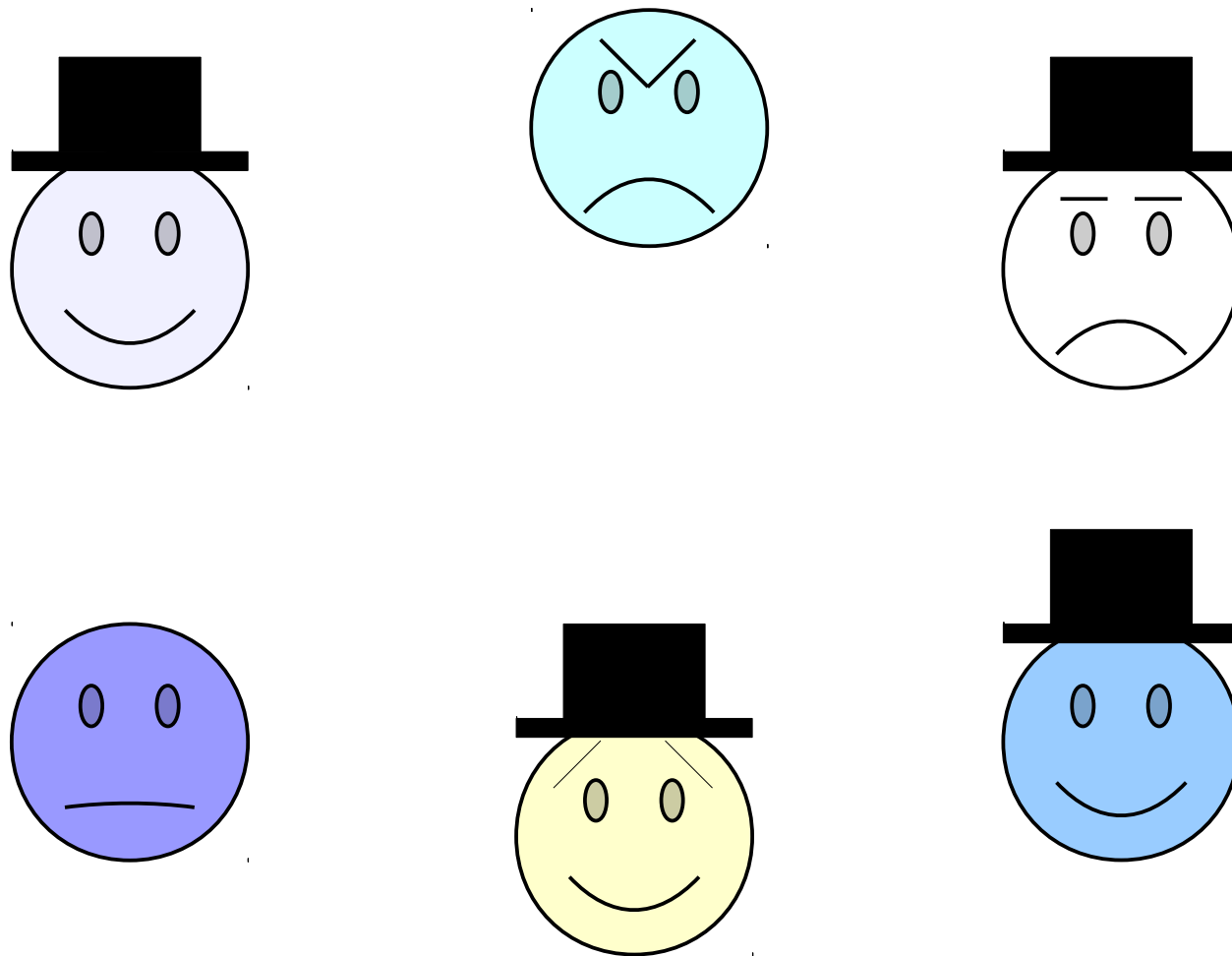
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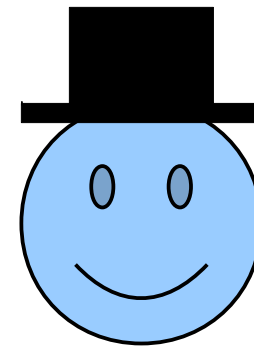
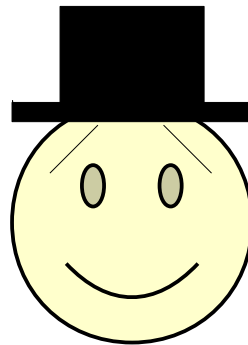
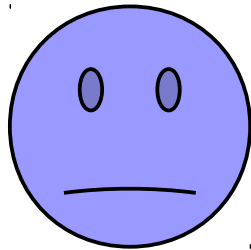
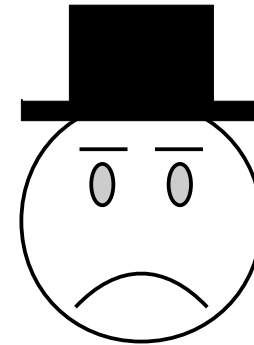
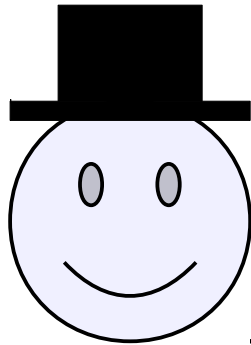
$\forall x. (\textit{Smiling}(x) \rightarrow \textit{WearingHat}(x))$



“Every smiling person wears a hat.”

$\forall x. (Smiling(x) \wedge WearingHat(x))$

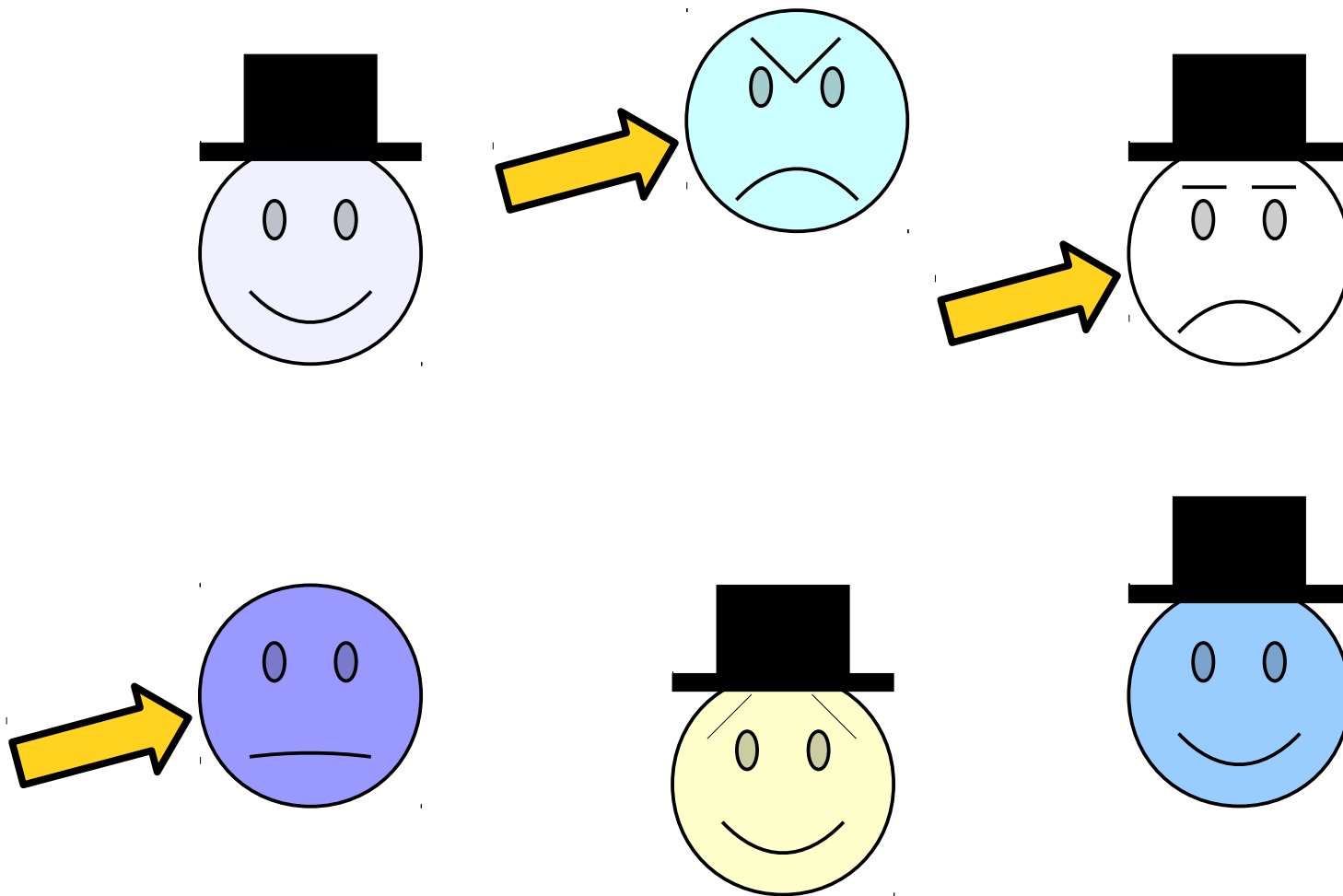
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“Every smiling person wears a hat.” *True*

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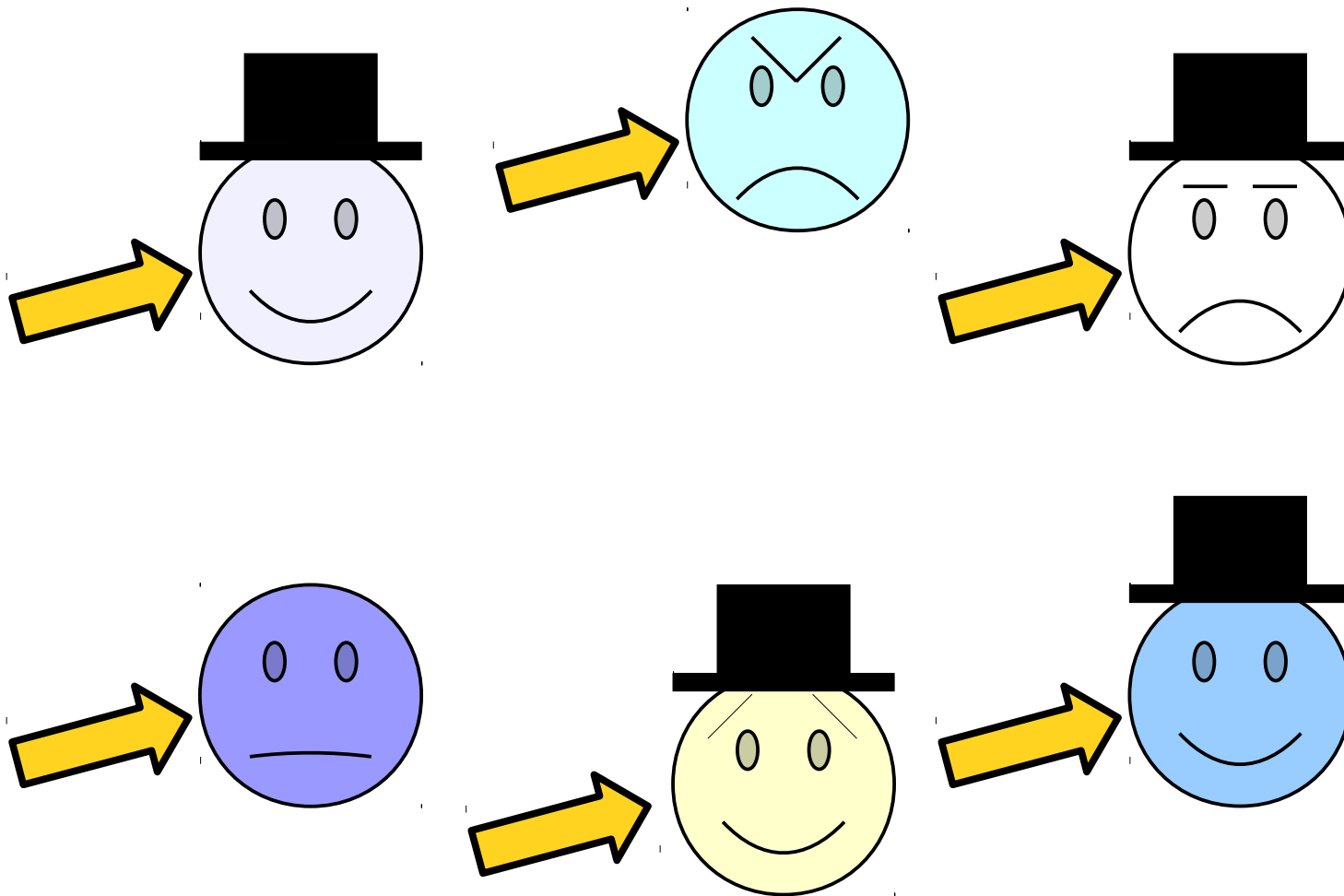
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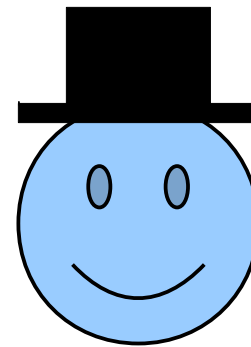
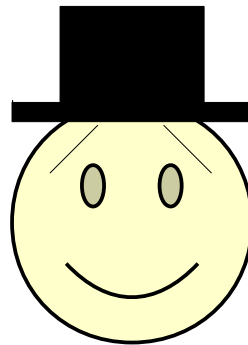
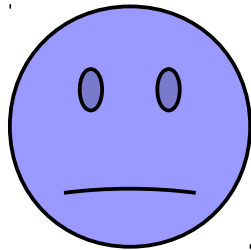
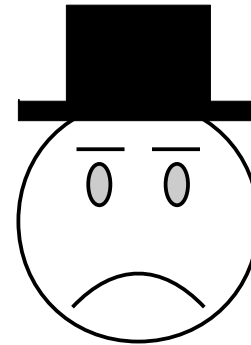
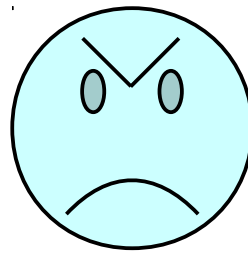
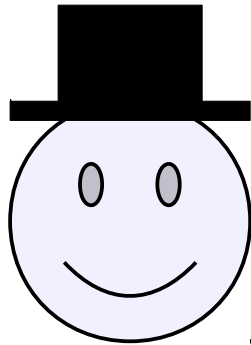
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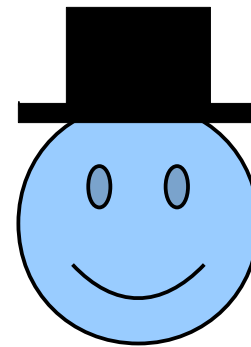
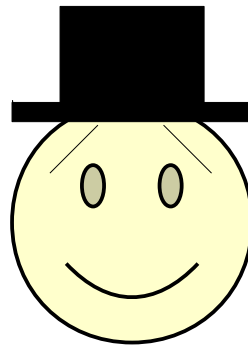
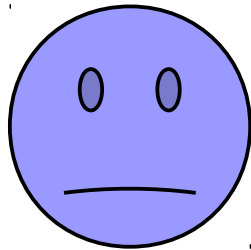
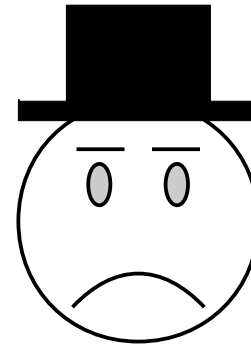
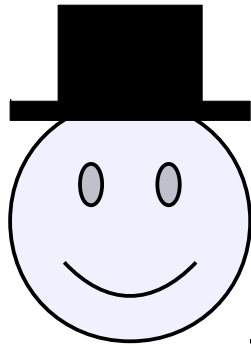
$\forall x. (Smiling(x) \rightarrow WearingHat(x))$ **True**



“Every smiling person wears a hat.” **True**

$\forall x. (Smiling(x) \wedge WearingHat(x))$ **False**

$\forall x. (Smiling(x) \rightarrow WearingHat(x))$ **True**



“Every smiling person wears a hat.” **True**

~~$\forall x. (Smiling(x) \wedge WearingHat(x))$~~ **False**

$\forall x. (Smiling(x) \rightarrow WearingHat(x))$ **True**

“All P 's are Q 's”

translates as

$\forall x. (P(x) \rightarrow Q(x))$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x. (P(x) \rightarrow Q(x))$$

If x is a counterexample, it must have property P but not have property Q .

Good Pairings

- The \forall quantifier *usually* is paired with \rightarrow .

$$\forall x. (P(x) \rightarrow Q(x))$$

- The \exists quantifier *usually* is paired with \wedge .

$$\exists x. (P(x) \wedge Q(x))$$

- In the case of \forall , the \rightarrow connective prevents the statement from being *false* when speaking about some object you don't care about.
- In the case of \exists , the \wedge connective prevents the statement from being *true* when speaking about some object you don't care about.

Proofwriting Workshop

An Incorrect Set Theory Proof

Claim: If A , B , and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

⚠ **Incorrect!** ⚠ **Proof:** Consider arbitrary sets A , B , and C where $C \subseteq A \cup B$.

This means that every element of C is in either A or B . If all elements of C are in A , then $C \subseteq A$. Alternately, if everything in C is in B , then $C \subseteq B$. In either case, everything inside of C has to be contained in at least one of these sets, so the theorem is true. ■

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This is just repeating definitions and not making specific claims about specific variables.

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Why is this bad?

Claim: If A , B , and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

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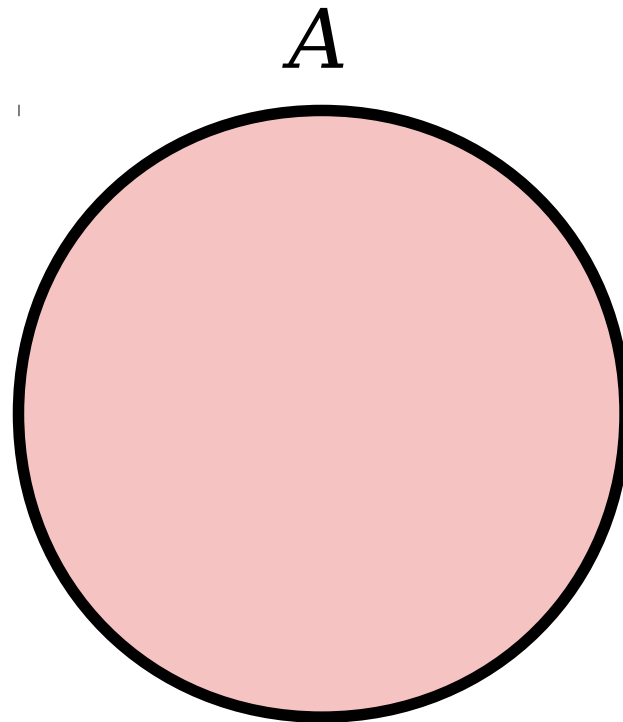
While this claim is true, it does not imply the theorem is true. In fact, this theorem is actually **false**.

Let's Draw Some Pictures!

Claim: If A , B , and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

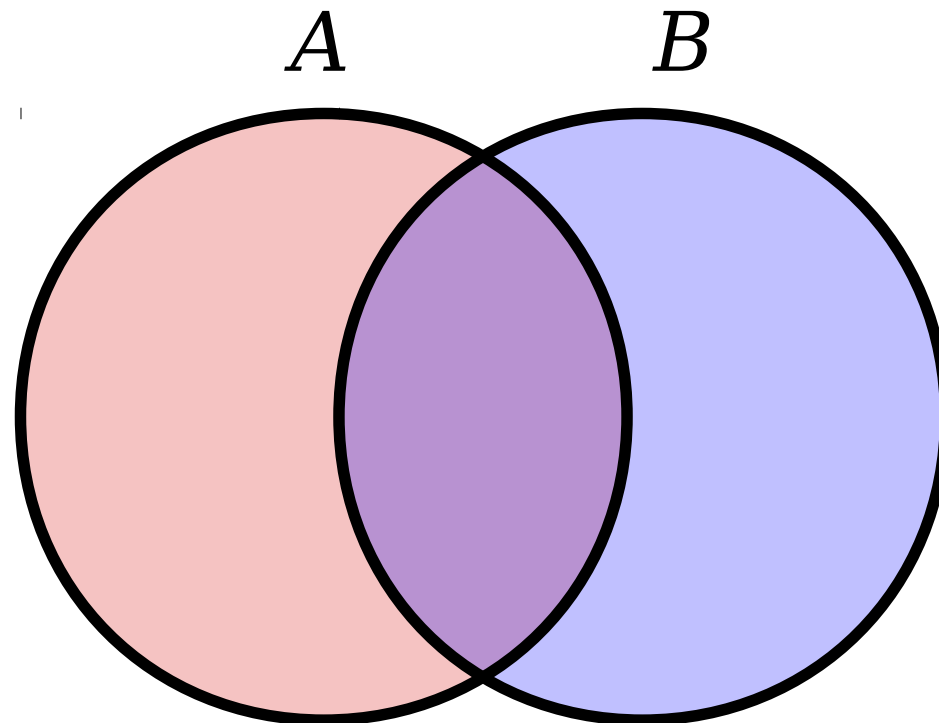
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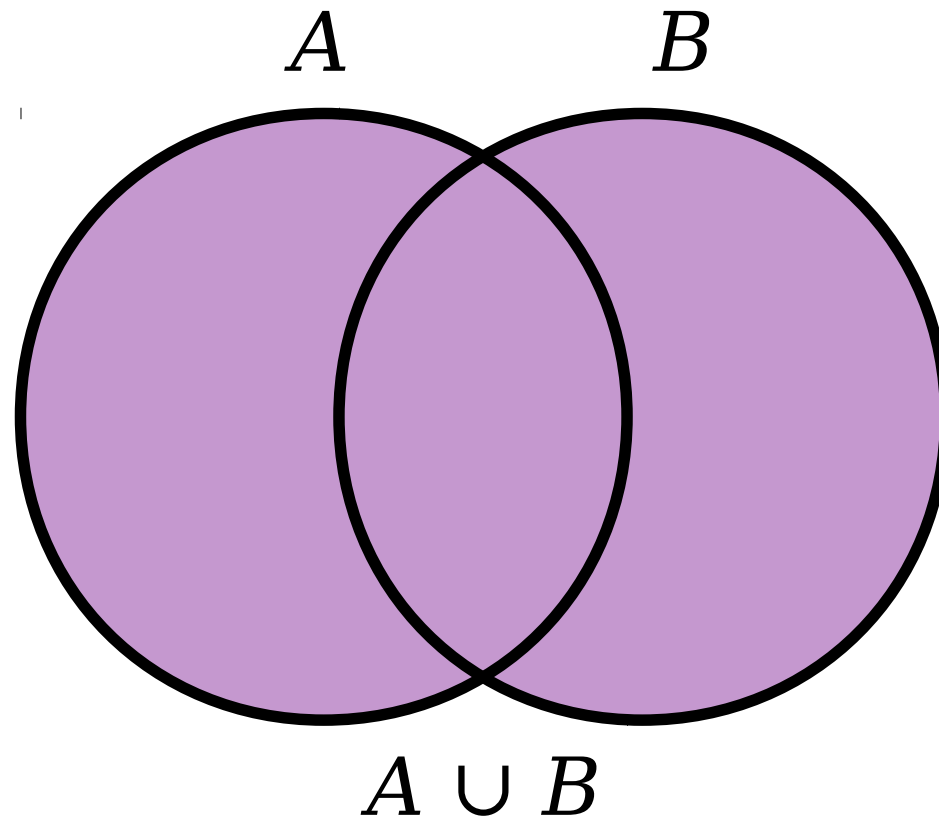
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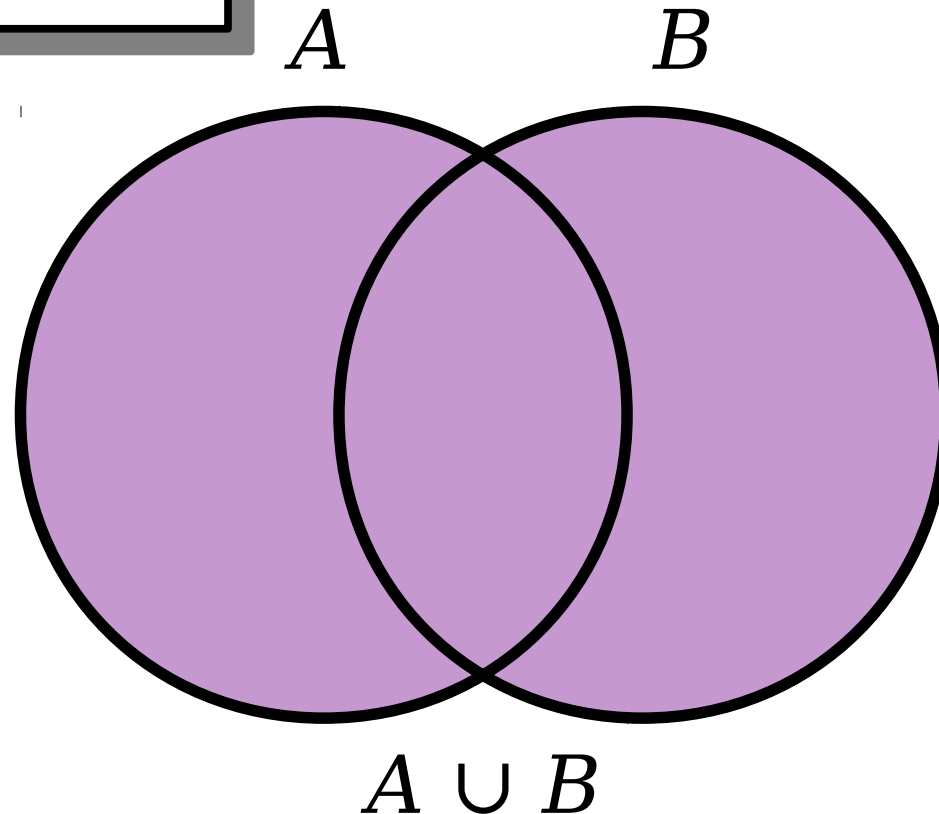
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Let's Draw Some Pictures!

Claim: If A , B , and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Recall the intuition of a subset being "something I can circle"

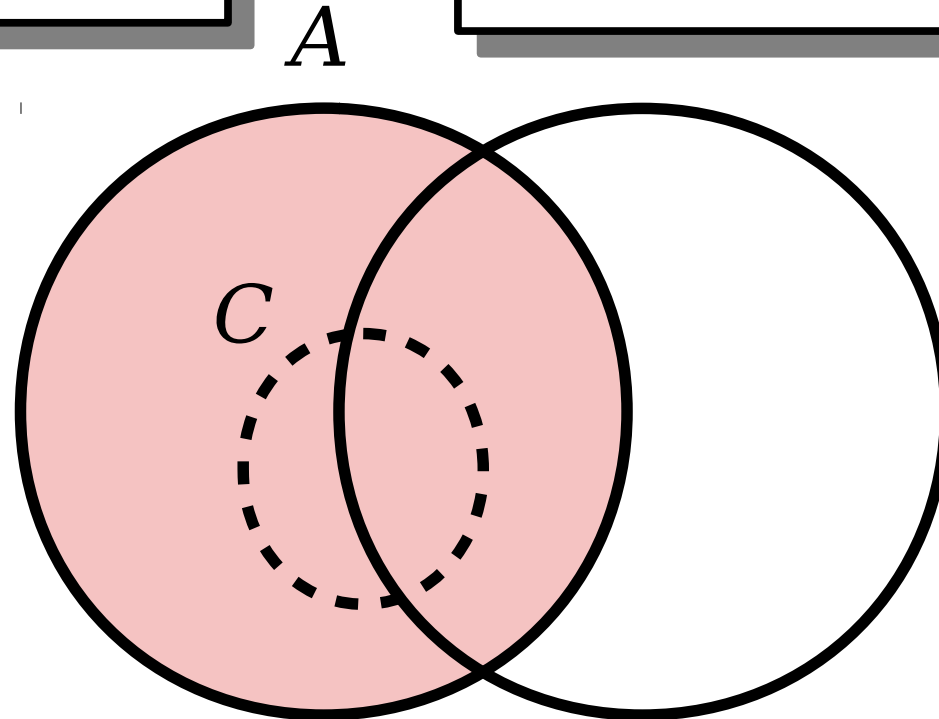


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Claim: If A , B , and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Recall the intuition of a subset being "something I can circle"

So $C \subseteq A$ would mean that C is something I can circle in this region.

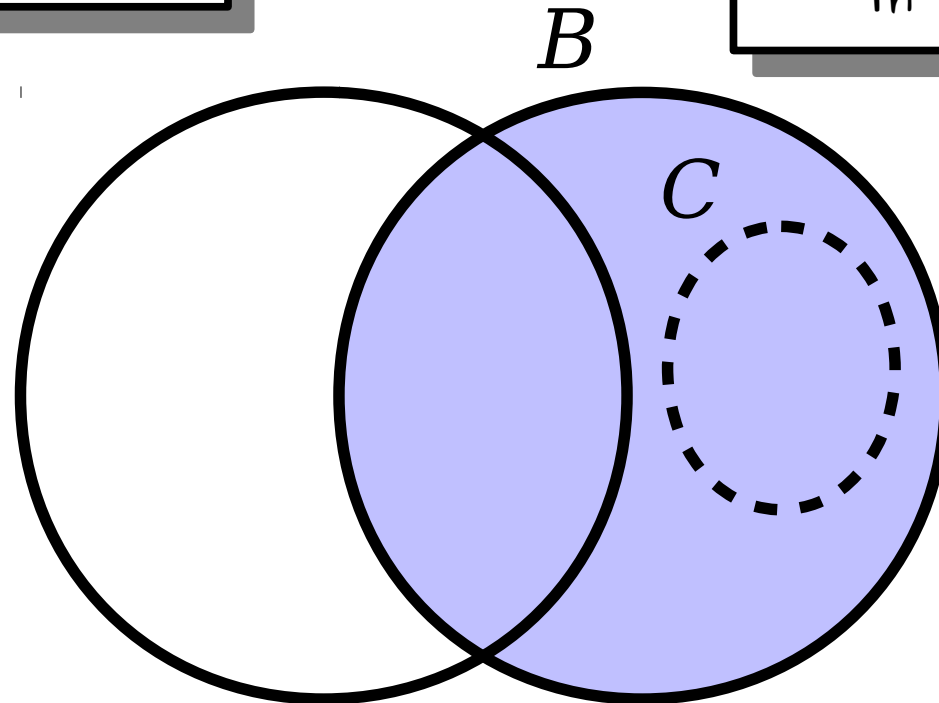


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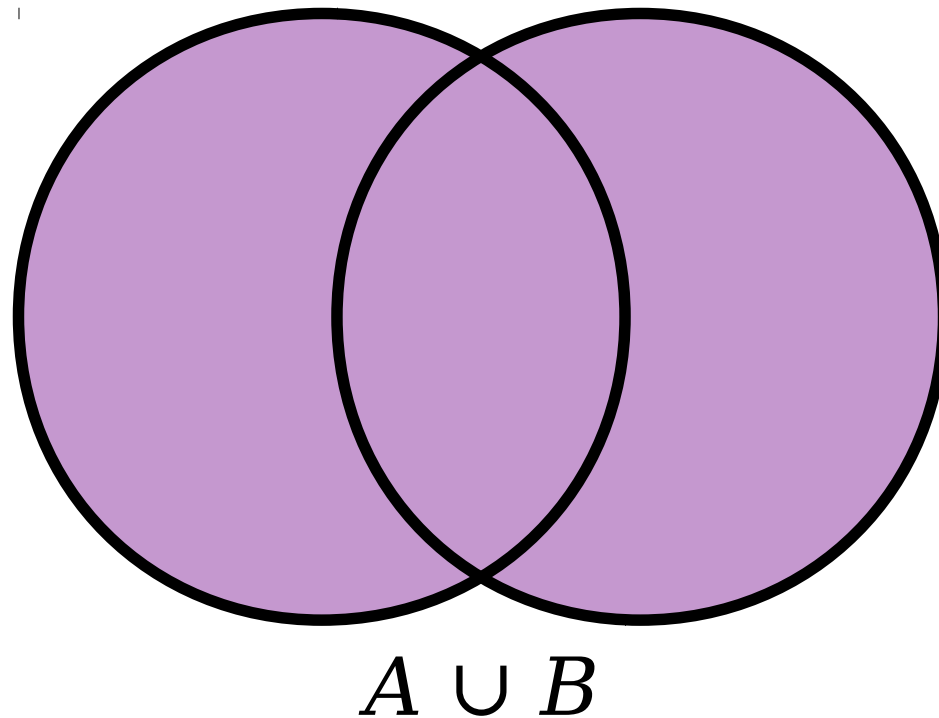
Recall the intuition of a subset being "something I can circle"

Likewise, $C \subseteq B$ would mean that C is something I can circle in this region.



Let's Draw Some Pictures!

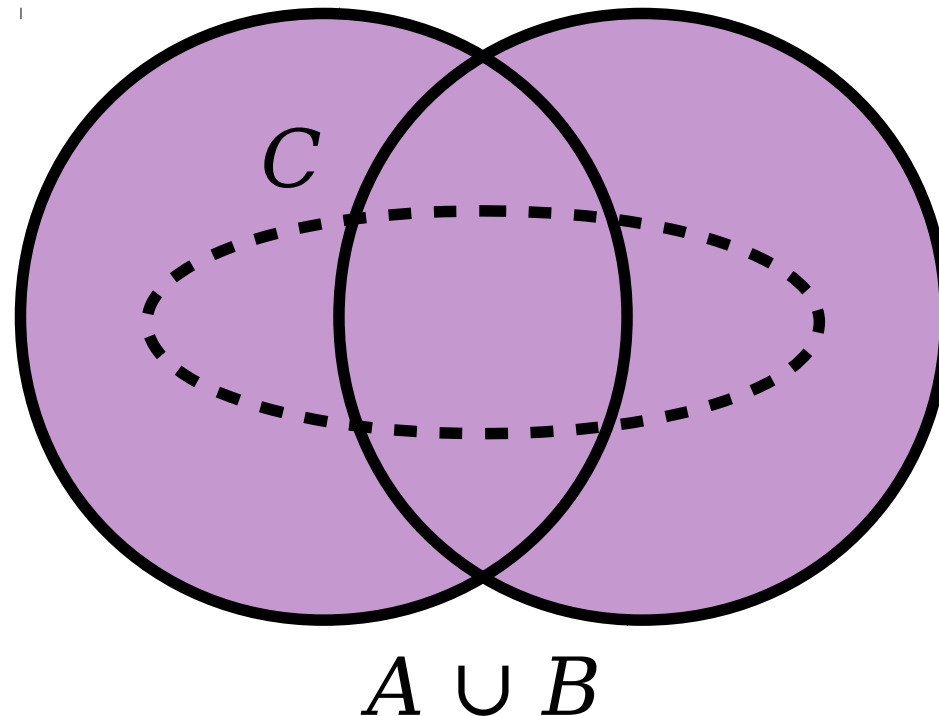
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Let's Draw Some Pictures!

Claim: If A , B , and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

But when I look at $A \cup B$, I can draw C as a circle containing elements from both A and B !

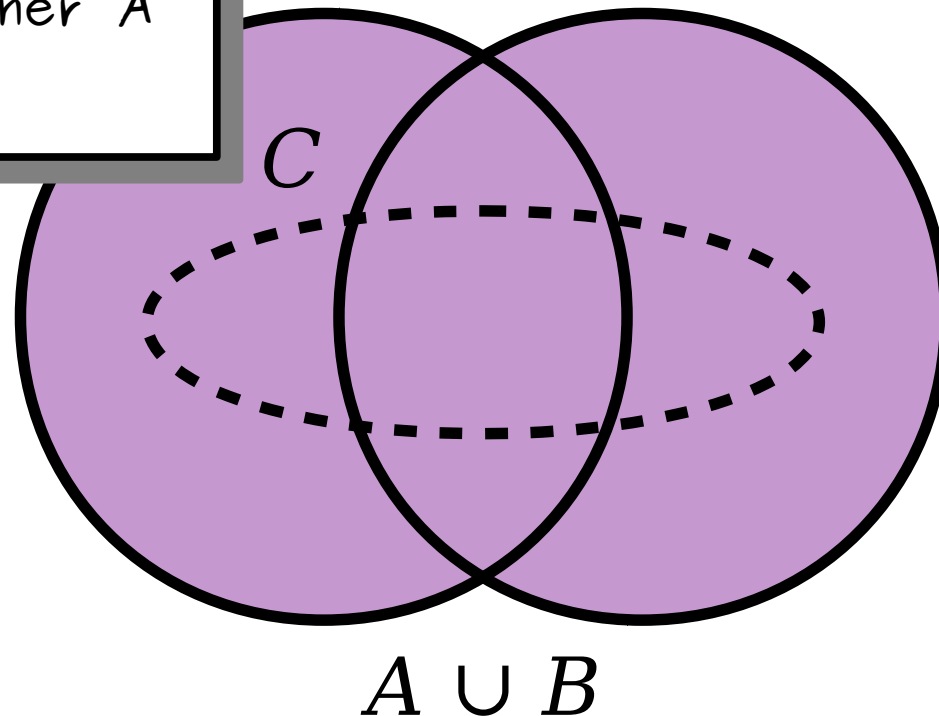


Let's Draw Some Pictures!

Claim: If A , B , and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

But when I look at $A \cup B$, I can draw C as a circle containing elements from both A and B !

Do you see why this circle is in neither A nor B ?

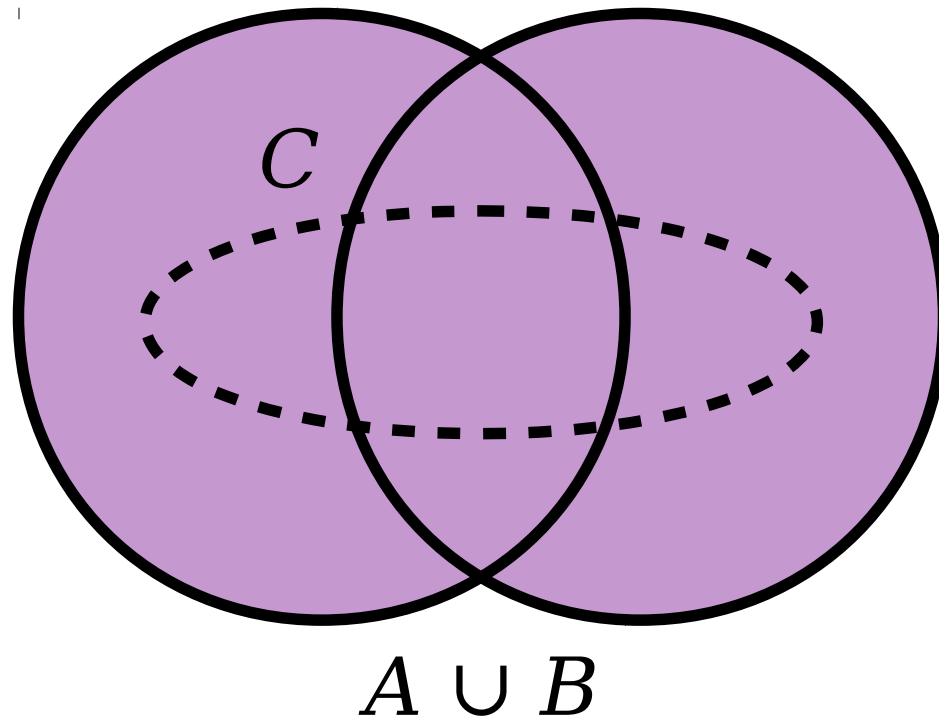


Let's Draw Some Pictures!

Claim: If A , B , and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Using this visual intuition, come up with a choice of sets A , B , and C that show this claim is false.

Respond at pollev.com/cs103



Proofs vs. Disproofs

- A ***proof*** is an argument that explains why some ***theorem*** is true.
- A ***disproof*** is an argument that explains why some ***claim*** is false.
- You've seen lots of examples of proofs.
What does a disproof look like?

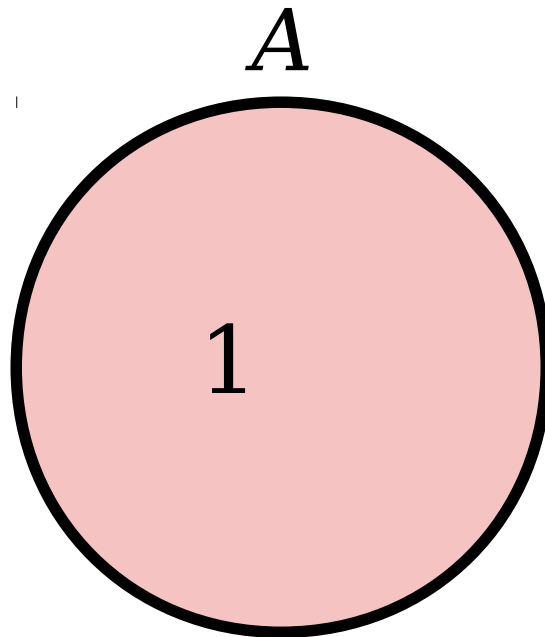
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Disproof: We will show that there are sets A , B , and C where $C \subseteq A \cup B$, but $C \not\subseteq A$ and $C \not\subseteq B$.

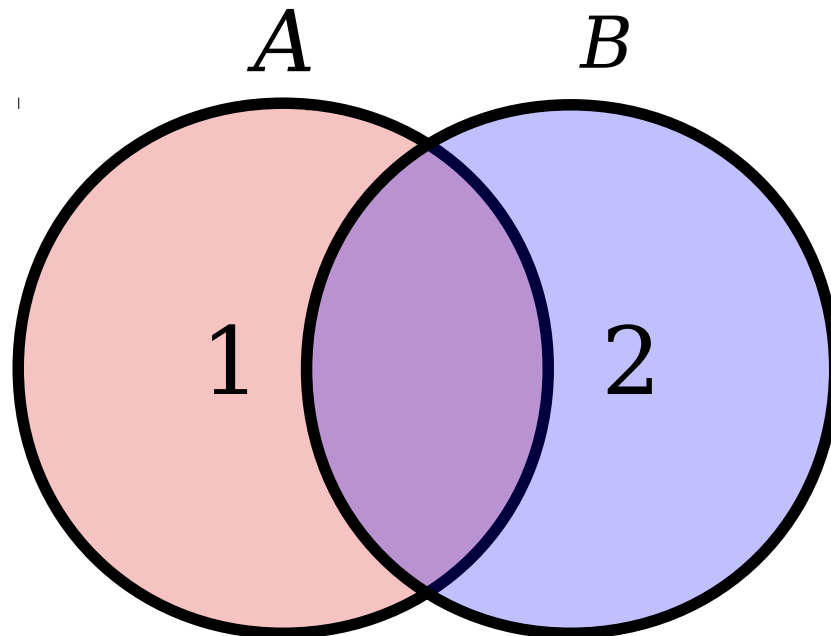
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Disproof: We will show that there are sets A , B , and C where $C \subseteq A \cup B$, but $C \not\subseteq A$ and $C \not\subseteq B$. Consider the sets $A = \{1\}$



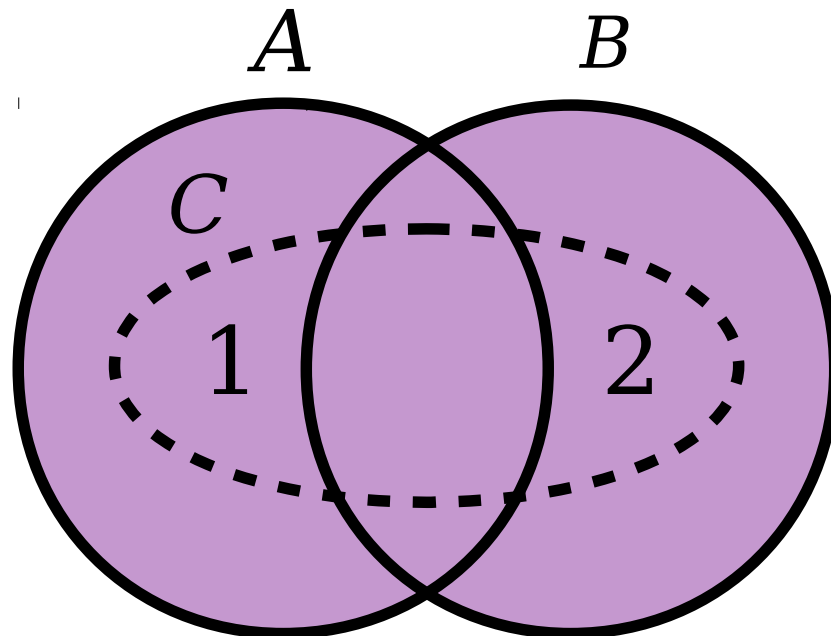
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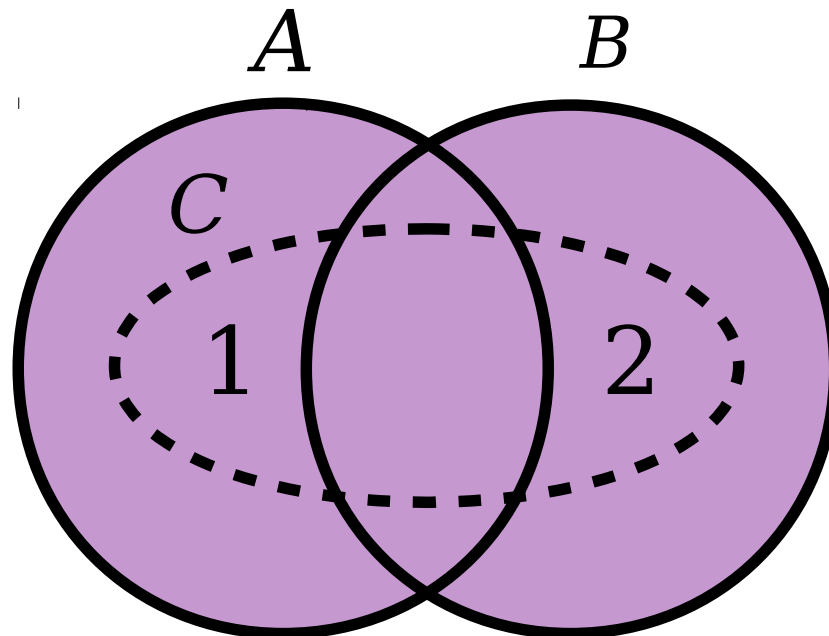
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Disproof: We will show that there are sets A , B , and C where $C \subseteq A \cup B$, but $C \not\subseteq A$ and $C \not\subseteq B$. Consider the sets $A = \{1\}$, $B = \{2\}$, and $C = \{1, 2\}$.



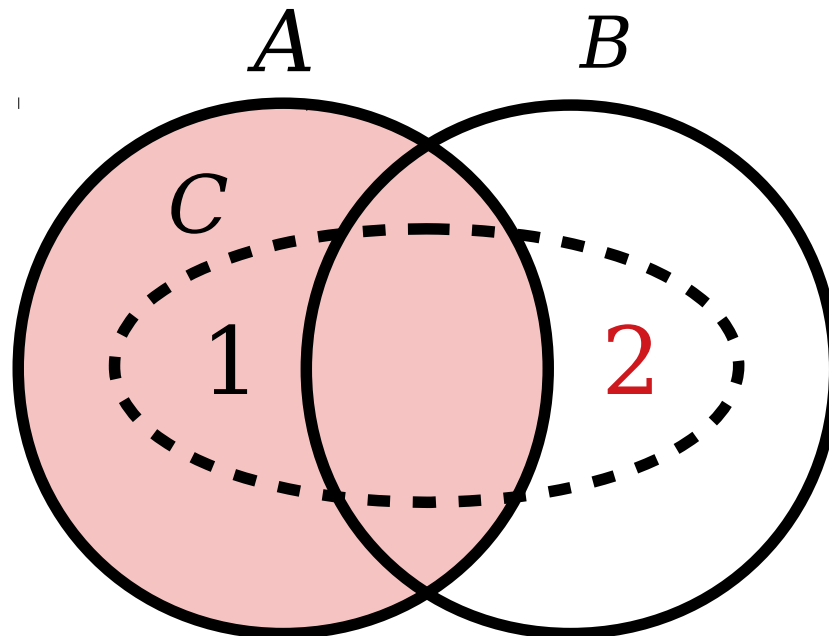
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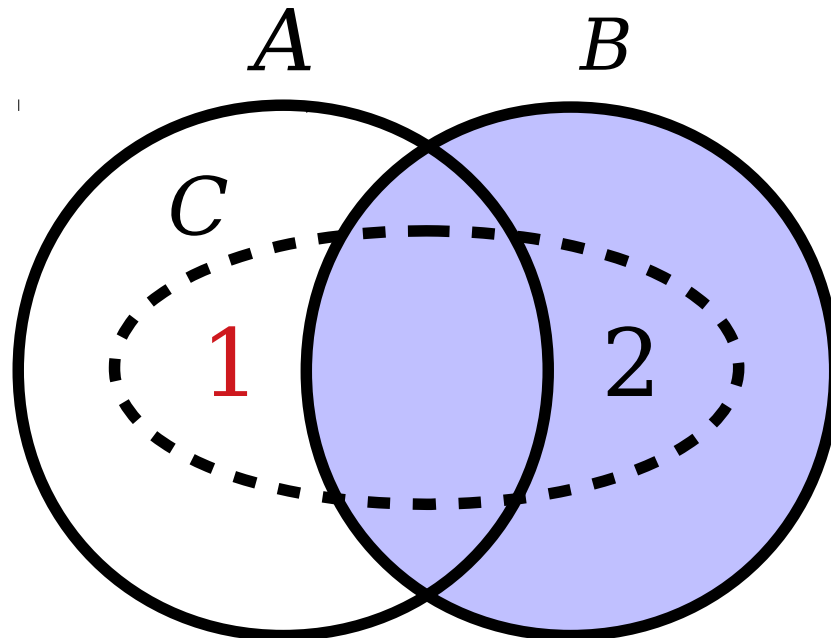
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Claim: If A , B , and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

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Thus we've found a set C which is a subset of $A \cup B$ but is not a subset of either A or B , which is what we needed to show. ■

Proofwriting Advice

- Be *very wary* of proofs that speak generally about “all objects” of a particular type.
 - As you’ve just seen, it’s easy to accidentally prove a false statement at this level of detail.
 - Making broad, high-level claims often indicates deeper logic errors or conceptual misunderstanding (like *code smell* but for proofs!)

Proofwriting Advice

A Very Good Idea: After you've written a draft of a proof, run through all of the points on the Proofwriting Checklist.

- This is a *great* exercise that you can do with a partner!

Proofs on Subsets

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

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Hold on, isn't this the claim we just disproved?

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

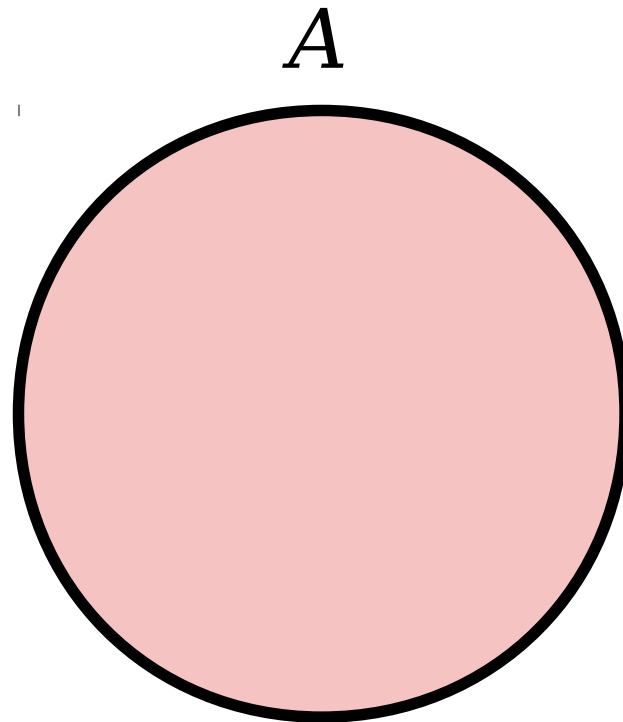
Notice that that's an intersection, not a union! It turns out that this claim is actually true.

Let's Draw Some Pictures!

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

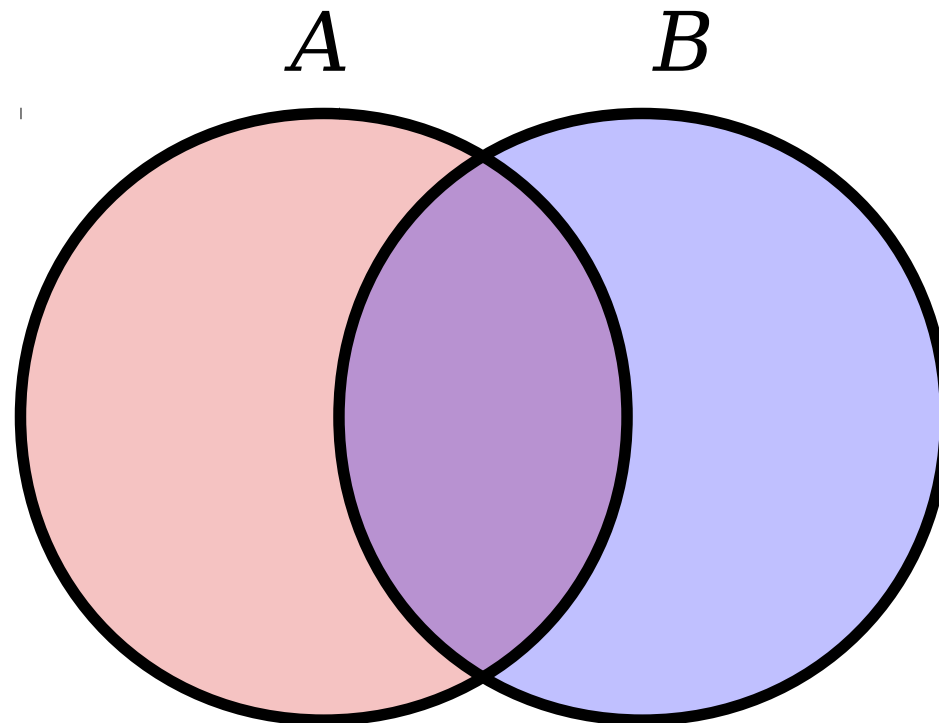
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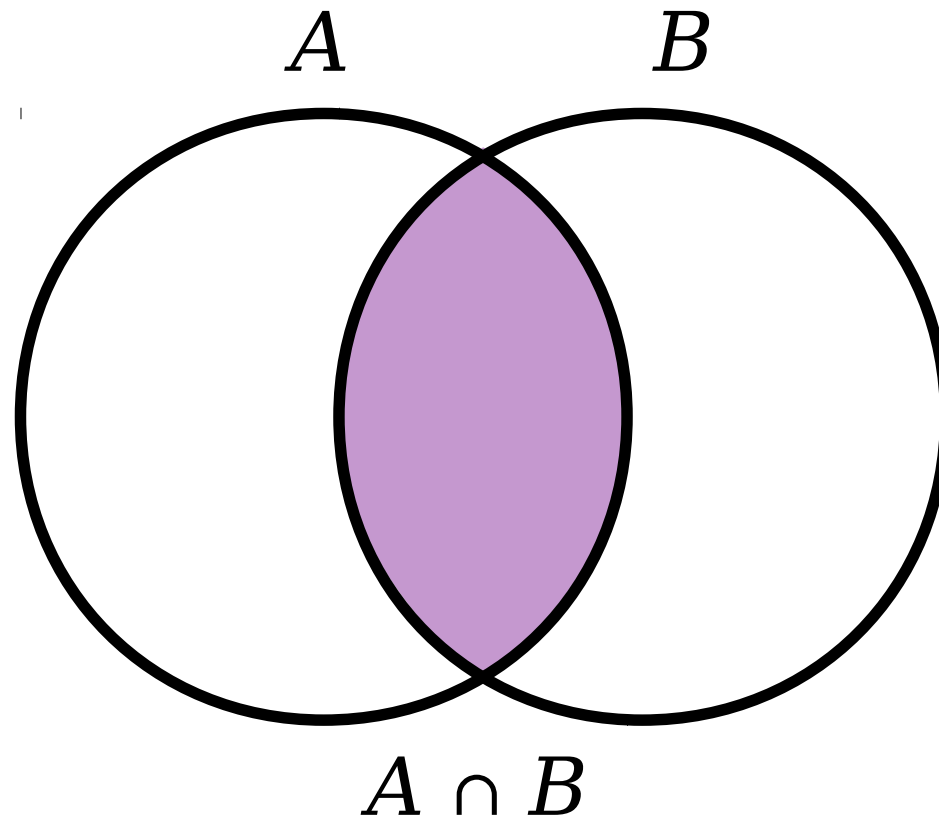
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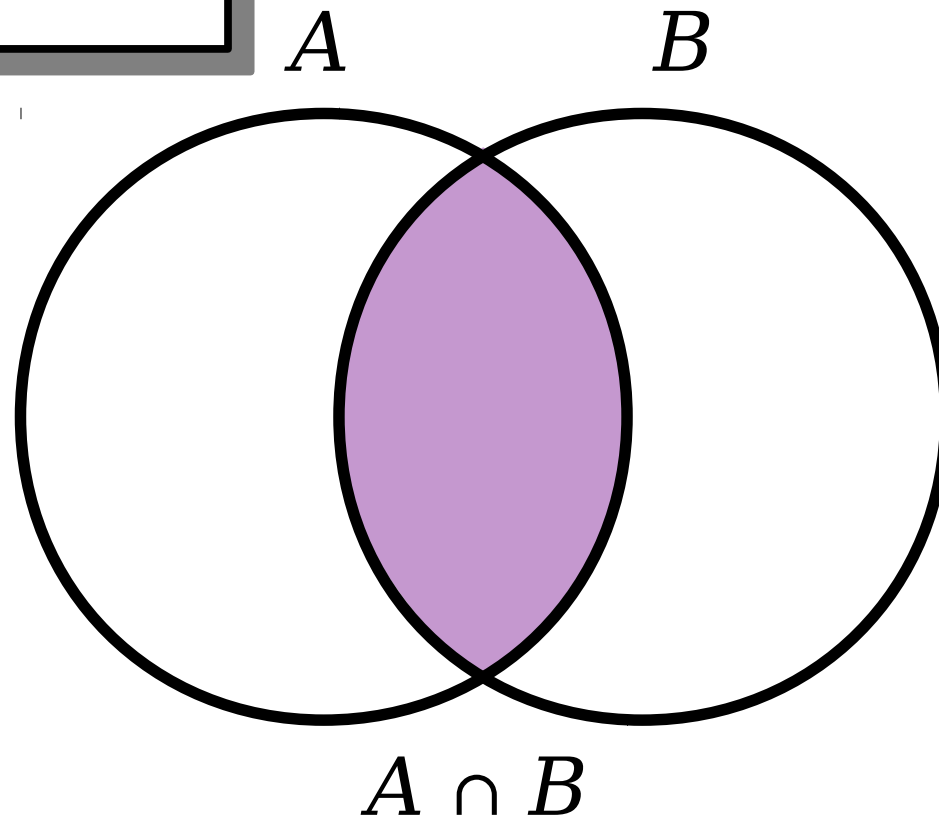
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Recall the intuition of a subset being "something I can circle"

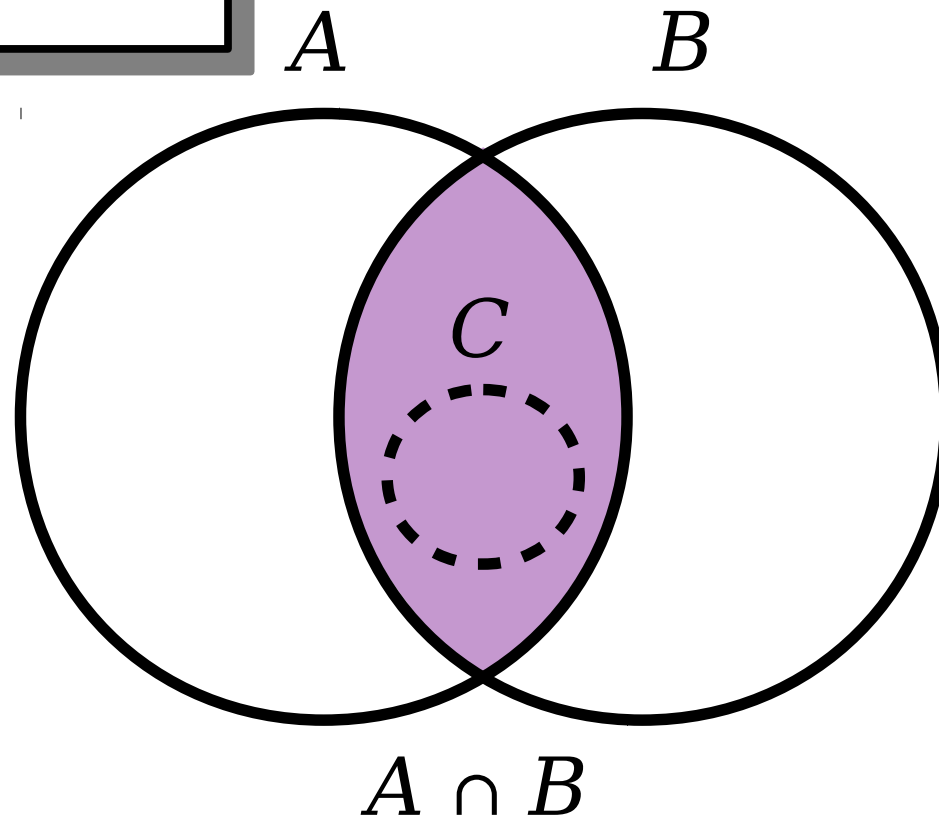


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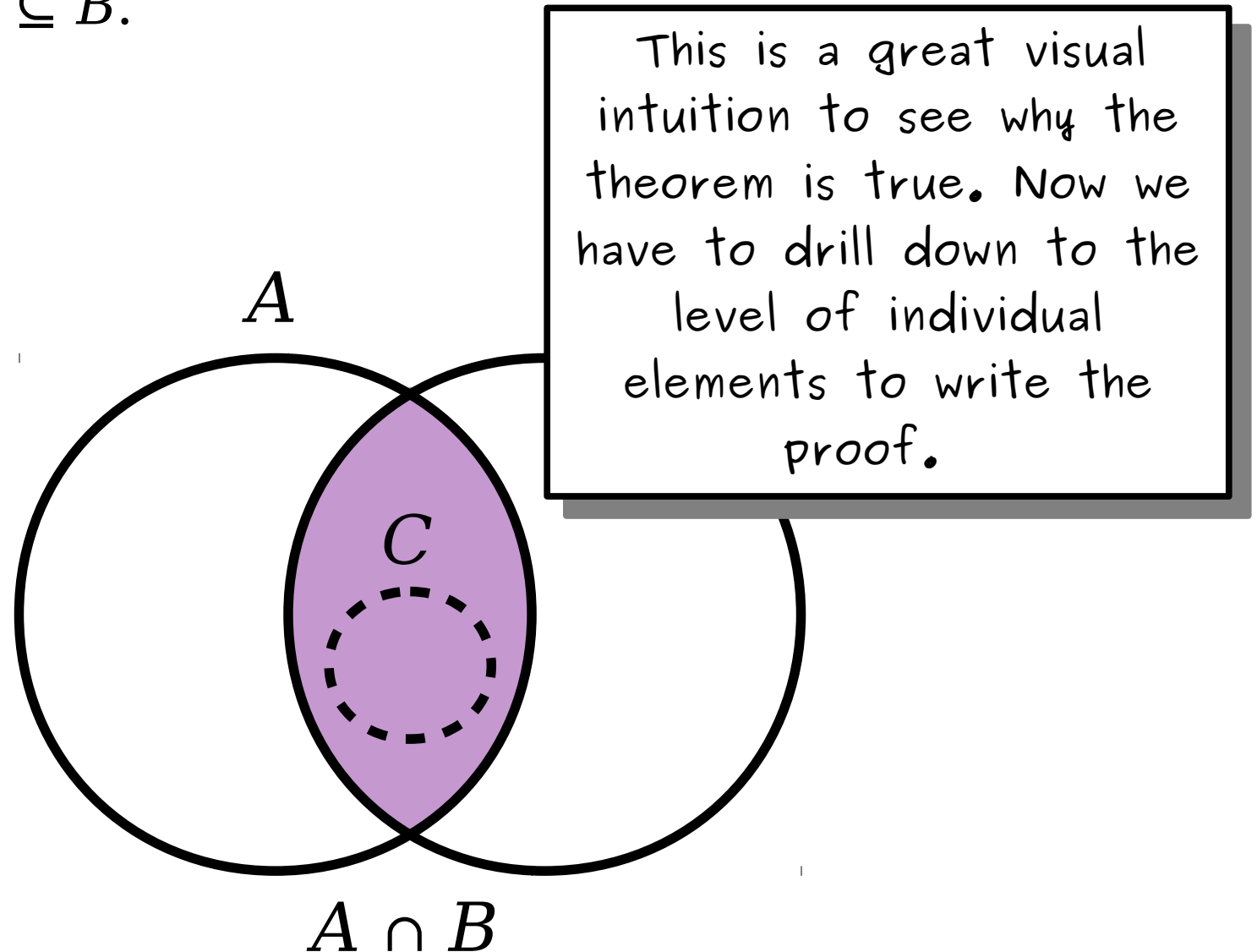
Recall the intuition of a subset being "something I can circle"

When I look at $A \cap B$, any circle I can draw in this region can be found in both A and B .



Let's Draw Some Pictures!

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.



Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

What We Need To Show

When confronted with a theorem to prove, the first step is to make sure you understand where you're starting and where you're going.

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A , B , and C are sets
- $C \subseteq A \cap B$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$

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What We're Assuming

- A , B , and C are sets
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What We Need To Show

- $C \subseteq A$ and $C \subseteq B$

A great proofwriting strategy is to **write down relevant definitions**. This gives you a better sense of what you need to prove and what tools you have at hand.

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A , B , and C are sets
- $C \subseteq A \cap B$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$

Before we start:

- What is the definition of subset?
- How do you prove that one set is a subset of another?
- If you know that one set is a subset of another, what can you conclude?

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A , B , and C are sets
- $C \subseteq A \cap B$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$

Definition: If S and T are sets, then $S \subseteq T$ when for every $x \in S$, we have $x \in T$.

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What We're Assuming

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What We Need To Show

- $C \subseteq A$ and $C \subseteq B$

Definition: If S and T are sets, then $S \subseteq T$ when for every $x \in S$, we have $x \in T$.

To prove that $S \subseteq T$:

Pick an arbitrary $x \in S$, then prove $x \in T$.

If you know that $S \subseteq T$:

If you have an $x \in S$, you can conclude $x \in T$.

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

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Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

This reading of the definition is usually helpful for unpacking this column!

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$

Definition: If S and T are sets, then $S \subseteq T$ when for every $x \in S$, we have $x \in T$.

To prove that $S \subseteq T$:

Pick an arbitrary $x \in S$, then prove $x \in T$.

If you know that $S \subseteq T$:

If you have an $x \in S$, you can conclude $x \in T$.

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A , B , and C are sets
- $C \subseteq A \cap B$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$

Our Tools

- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$

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How can we apply this general template to our specific problem?

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- Pick an $x \in C$, show that $x \in B$

Now we know that ultimately, we're going to have to do these two things. Let's see what tools we have that can get us here!

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What We're Assuming

- A , B , and C are sets
- $C \subseteq A \cap B$

This reading of the definition is usually helpful for unpacking this column!

that $x \in A$

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- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$
 - Pick an $x \in C$, show that $x \in A$
 - Pick an $x \in C$, show that $x \in B$

Before we continue:

- What is the definition of set intersection?

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A , B , and C are sets
- $C \subseteq A \cap B$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$
 - Pick an $x \in C$, show that $x \in A$

Definition: The set $S \cap T$ is the set where, for any x :
 $x \in S \cap T$ when $x \in S$ and $x \in T$

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

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- A , B , and C are sets
- $C \subseteq A \cap B$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$
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Definition: The set $S \cap T$ is the set where, for any x :
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To prove that $x \in S \cap T$:

Prove both that $x \in S$ and that $x \in T$.

If you know that $x \in S \cap T$:

You can conclude both that $x \in S$ and that $x \in T$.

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A , B , and C are sets
- $C \subseteq A \cap B$

This is the one we want!

- Pick an $x \in C$, show that $x \in A$

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Our Tools

- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$.
- If you know that $S \subseteq T$ and you have an $x \in S$, you can conclude $x \in T$.
- If you know that $x \in S \cap T$, we can conclude that $x \in S$ and $x \in T$.

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$
- Pick an $x \in C$, show that $x \in A$
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What We Need To Show

- $C \subseteq A$ and $C \subseteq B$
 - Pick an $x \in C$, show that $x \in A$
 - Pick an $x \in C$, show that $x \in B$

Let's go and try and do the proof with what we've got here!

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

Rough Outline

- Assume $C \subseteq A \cap B$

Relevant Definitions

- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$
- If you know that $S \subseteq T$ and you have an $x \in S$, you can conclude $x \in T$.
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Rough Outline

- Assume $C \subseteq A \cap B$
- Proving $C \subseteq A$
 - Pick an $x \in C$
- Conclude $x \in A$

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- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$
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Rough Outline

- Assume $C \subseteq A \cap B$
- Proving $C \subseteq A$
 - Pick an $x \in C$

What goes here?

- Conclude $x \in A$

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- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$
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- Assume $C \subseteq A \cap B$
- Proving $C \subseteq A$
 - Pick an $x \in C$
 - $x \in A \cap B$
 - $x \in A$ and $x \in B$
 - Conclude $x \in A$

We also need to prove that
 $C \subseteq B$.

Notice that if you take the outline here and literally swap the variable A for the variable B , you get a working proof.

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

Rough Outline

- Assume $C \subseteq B \cap A$
- Proving $C \subseteq B$
 - Pick an $x \in C$
 - $x \in B \cap A$
 - $x \in B$ and $x \in A$
 - Conclude $x \in B$

In a case like this where your proof would have two completely symmetric branches, it's fine to write up just one and say "**by symmetry**, [the other branch] is also true."

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

Rough Outline

- Assume $C \subseteq A \cap B$
- Proving $C \subseteq A$
 - Pick an $x \in C$
 - $x \in A \cap B$
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Try it yourself: Take a few minutes and write up a proof of the theorem using this outline.

Then share your proof with your neighbors and critique each other!

Respond at pollev.com/cs103

Theorem: If A , B , and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

Proof: Let A , B , and C be arbitrary sets where $C \subseteq A \cap B$. We need to show that $C \subseteq A$ and $C \subseteq B$. Because the roles of A and B in this proof are symmetric, we can just prove that $C \subseteq A$.

Choose any element $x \in C$. Since $C \subseteq A \cap B$, we know that $x \in A \cap B$. This tells us that $x \in A$ and $x \in B$. In particular, this means that $x \in A$, thus completing the proof. ■

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Are you clearly stating what you're assuming and what you're trying to prove?

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Are you making specific claims about specific variables? Your proof should NOT have statements of the form "every element of C ".

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- Are all variables properly introduced and scoped? You should be able to point at every variable and say that it is either:
- 1) an arbitrarily chosen value - owned by the reader
 - 2) an existentially instantiated value - owned by no one
 - 3) an explicitly chosen value - owned by you (the proof writer)

Next Time

- ***First-Order Translations***
 - How do we translate from English into first-order logic?
- ***Quantifier Orderings***
 - How do you select the order of quantifiers in first-order logic formulas?
- ***Negating Formulas***
 - How do you mechanically determine the negation of a first-order formula?
- ***Expressing Uniqueness***
 - How do we say there's just one object of a certain type?